Today: classification of semisimple Lie algebras

Recall: A (f.d., complex) Lie algebra is a $\mathbb{C}$-v.s. $\mathfrak{g}$ w/ a map $[\cdot,\cdot]: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is bilinear, skew-symmetric, and satisfies the Jacobi identity

Root space decomposition: If $\mathfrak{g}$ is semisimple (direct sum of simples) w/ $\underbrace{\text{Cartan subalg}}_{\text{max'l abelian, } \mathrm{ad}_{\mathfrak{h}} \text{ diagonalizable}}$ $\mathfrak{h}$, then (as a v.s.),

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \mathrm{ad}_H(X) = \alpha(H) X \ \forall H \in \mathfrak{h} \}$

Properties: Φ is a root system, root spaces are 1-dim., and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$.

Each $\alpha \in \Phi^+$ is assoc. to a copy of $\mathfrak{sl}_2$, span$\{e_\alpha, f_\alpha, h_\alpha\}$

Example:

$$\mathfrak{g} = \mathfrak{sl}_n = \{\text{traceless } n \times n \text{ matrices}\}$$

Cartan subalg.:

$$\mathfrak{h} = \Bigg\{ \text{diag. matrices } \underbrace{\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}}_{H} \;\Bigg|\; z_1 + \cdots + z_n = 0 \Bigg\}$$

$$\mathfrak{g} = \mathfrak{h} \oplus \operatorname{span}\left\{ E_{ij} = \begin{array}{cc} \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} & i \\ \quad\; j & \end{array} \;\middle|\; i \neq j \right\}$$

$$[H, E_{ij}] = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} - \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$$

$$= z_i E_{ij} - z_j E_{ij}$$

$$= (z_i - z_j) E_{ij}$$

$$= \alpha_{ij}(H) E_{ij} \quad \text{where } \alpha_{ij}\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = z_i - z_j$$

The α_{ij} are exactly the A_{n-1} roots

$$\{ e_i - e_j \mid 1 \leq i, j \leq n,\ i \neq j \},$$

interpreting e_i as the functional $e_i\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = z_i$

potentially confusing notation

If $i<j$, $\alpha=\alpha_{ij}$, we have

$$e_\alpha = E_{ij}\ ,\quad f_\alpha = E_{ji}\ ,\quad h_\alpha = \begin{array}{c c} \left(\begin{array}{cc} 1 & \\ & -1 \end{array}\right) & \begin{array}{c} i \\ j \end{array} \\ \begin{array}{cc} i & j \end{array} & \end{array}$$

If $n=3$, we have

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$$\Phi = \{\overbrace{e_1-e_2}^{\alpha_1},\ \overbrace{e_1-e_3}^{\alpha_1+\alpha_2},\ \overbrace{e_2-e_3}^{\alpha_2},\ e_2-e_1,\ e_3-e_1,\ e_3-e_2\}$$

$$\underbrace{e_1-e_2,\ e_1-e_3,\ e_2-e_3}_{\Phi^+}$$

Root spaces:

$$\begin{bmatrix} \mathfrak{h} & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} \\ \mathfrak{g}_{-\alpha_1} & \mathfrak{h} & \mathfrak{g}_{\alpha_2} \\ \mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & \mathfrak{h} \end{bmatrix}$$

$$e_{\alpha_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad f_{\alpha_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad h_{\alpha_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad f_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad h_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Thm 54 (Killing-Cartan classification):

Complex semisimple Lie algebras are in correspondence w/ root systems (and therefore Dynkin diagrams).

Pf: The v.s. and bracket structure are det'd by the root space decomp. and other properties

Up to taking direct sums, these are:

Classical:	dimensions
$A_n : \mathfrak{sl}_{n+1}$	$n(n+2)$
$B_n : \mathfrak{so}_{2n+1}$ (special orthogonal)	$2n^2+n$
$C_n : \mathfrak{sp}_{2n}$ (symplectic)	$2n^2+n$
$D_n : \mathfrak{so}_{2n}$ $(n>1)$	$2n^2-n$

Exceptional:

Fulton-Harris Ch. 22

$$E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

dims: 78, 133, 248, 52, 14

Symplectic Lie algebras (type C)

$$\mathfrak{sp}_{2n} = \{ X \in \mathfrak{gl}_{2n} \mid JX + X^T J = 0 \}$$

where
$$J = \left[\begin{array}{c|c} 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & 0 \end{array}\right] \begin{array}{l} \}\, n \\ \}\, n \end{array}$$

Class activity (if time):

If $X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, what does this say about A, B, C, D?

(B, C symmetric, $D = -A^T$)

$$\underset{\substack{\text{Cartan}\\\text{subalg.}}}{\mathfrak{h}} = \mathfrak{sp}_{2n} \cap \{\text{diag matrices}\} = \left\{ \left[\begin{array}{c|c} \begin{matrix} z_1 & & \\ & \ddots & \\ & & z_n \end{matrix} & \\ \hline & \begin{matrix} -z_1 & & \\ & \ddots & \\ & & -z_n \end{matrix} \end{array}\right] \right\}$$

Roots: $\{\pm e_i \pm e_j, \pm 2e_i\}$ where $e_i(\cdots) = z_i$

Embedded $\mathfrak{sl}_2$'s:

$H_i := E_{ii} - E_{\underline{i}\,\underline{i}}$

$\underline{i} := i + n$

α	e_α	f_α	h_α
$e_i - e_j$	$E_{ij} - E_{\underline{j}\,\underline{i}}$	$E_{ji} - E_{\underline{i}\,\underline{j}}$	$H_i - H_j$
$e_i + e_j$	$E_{i,\underline{j}} + E_{j,\underline{i}}$	$E_{\underline{i},j} + E_{\underline{j},i}$	$H_i + H_j$
$2e_i$	$E_{i,\underline{i}}$	$E_{\underline{i},i}$	H_i

Special orthogonal Lie algebra (type B (odd) and D (even))

$$\mathfrak{so}_{2n(+1)} = \{ X \in \mathfrak{gl}_{2n(+1)} \mid BX + X^T B = 0 \}$$

where
$$B = \left[\begin{array}{c|c|c} 0 & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & 0 \\ \hline \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} & 0 & 0 \\ \hline 0 & 0 & (1) \end{array}\right] \begin{array}{l} \}\, n \\ \}\, n \\ \}\text{— for } \mathfrak{so}_{2n+1} \end{array}$$

If
$$X = \left[\begin{array}{c|c|c} A & B & (E) \\ \hline C & D & (F) \\ \hline (G) & (H) & (J) \end{array}\right] \in \mathfrak{so}_{2n} \quad (\in \mathfrak{so}_{2n+1})$$

then B, C skew-symmetric, $D = -A^T$, $(E = -H^T,\ F = -G^T,\ J = 0)$

$$\mathfrak{h} = \mathfrak{so}_{2n(+1)} \cap \{\text{diag matrices}\} = \underbrace{\left\{ \left[\begin{array}{c|c|c} \begin{smallmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{smallmatrix} & & \\ \hline & \begin{smallmatrix} -z_1 & & \\ & \ddots & \\ & & -z_n \end{smallmatrix} & \\ \hline & & (0) \end{array}\right] \right\}}$$

Cartan subalg.

Roots: $\{ \pm e_i \pm e_j, (\pm e_i) \}$ where $e_i(\text{--}) = z_i$

Embedded $\mathfrak{sl}_2$'s:

$H_i := E_{ii} - E_{\underline{i}\,\underline{i}}$

$\underline{i} := i + n$

α	e_α	f_α	h_α
$e_i - e_j$	$E_{ij} - E_{\underline{j}\,\underline{i}}$	$E_{ji} - E_{\underline{i}\,\underline{j}}$	$H_i - H_j$
$e_i + e_j$	$E_{i,\underline{j}} - E_{j,\underline{i}}$	$E_{\underline{i},j} - E_{\underline{j},i}$	$H_i + H_j$
(e_i	$E_{i,2n+1} - E_{2n+1,\underline{i}}$	$E_{\underline{i},2n+1} - E_{2n+1,i}$	$2H_i$)