Reminder: HW3 due this Wed.

Recall:

a) $\mathfrak{sl}_2(\mathbb{C})$ irreps

- det'd by their largest h-evalue k $\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$
- k nonneg int., dim of irrep. is $2k+1$
- e/f raise/lower into higher/lower h-espaces

b) root systems

- finite subsets of Euclidean space closed under a reflection gp. and satisfying integrality conds. ("Weyl gp")
- lots of structure (simple root, positive roots, etc.)
- Dynkin classification: $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$

Def 49: A (f.d., complex) <u>Lie algebra</u> is a $\mathbb{C}$-v.s. $\mathfrak{g}$ w/ a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ s.t.

a) $[\cdot,\cdot]$ is bilinear

b) Skew-symmetry: $[X,Y] = -[Y,X] \quad \forall X, Y \in \mathfrak{g}$

c) Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \forall X, Y, Z \in \mathfrak{g}$

Remark:

- Condition b) implies $[X,X]=0$
- $[X,Y] = XY - YX$ satisfies these properties e.g.

$$
\begin{aligned}
&[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] \\
&= X(YZ-ZY) - (YZ-ZY)X \\
&\quad + Y(ZX-XZ) - (ZX-XZ)Y \\
&\quad + Z(XY-YX) - (XY-YX)Z \\
&= 0
\end{aligned}
$$

Def 50:

a) An <u>ideal</u> of a Lie alg. $\mathfrak{g}$ is a $\underbrace{\text{Lie subalg.}}_{}$ $\mathfrak{h} \leq \mathfrak{g}$ s.t. s.t. $[X,H] \in \mathfrak{h}$ $\forall X \in \mathfrak{g}$, $H \in \mathfrak{h}$

subspace closed under $[\cdot,\cdot]$

b) $\mathfrak{g}$ is <u>indecomposible</u> if its only ideals are $\mathfrak{g}$ and $\{0\}$, and <u>simple</u> if it is indecomp. and has dim ≥ 2.

c) $\mathfrak{g}$ is <u>reductive</u> if it's a direct sum of indecomposible Lie algebras, and <u>semisimple</u> if it's a direct sum of simple Lie algebras

- $\mathfrak{sl}_n$ is semisimple ← running example
- $\mathfrak{gl}_n$ is reductive

Def / Prop 51: A Cartan subalgebra of $\mathfrak{g}$ is a subspace $\mathfrak{h} \leq \mathfrak{g}$ s.t.

a) $\mathfrak{h}$ is abelian: $[H, H'] = 0 \ \forall H, H' \in \mathfrak{h}$

b) If $\mathfrak{h} \subsetneq \mathfrak{h}' \leq \mathfrak{g}$, then $\mathfrak{h}'$ is not abelian

c) For all $H \in \mathfrak{h}$, $ad_H : X \mapsto [H, X]$ is diagonalizable.

previously called this $ad(H)$

Every semisimple Lie algebra has a Cartan subalg.

The rank of $\mathfrak{g}$ is the dimension of any Cartan subalg.

Def 52: Fix a Lie algebra $\mathfrak{g}$ w/ Cartan subalg. $\mathfrak{h}$.

A root of $\mathfrak{g}$ (relative to $\mathfrak{h}$) is a nonzero linear functional $\alpha : \mathfrak{h} \longrightarrow \mathbb{C}$ s.t. $\exists X \in \mathfrak{g} \setminus \{0\}$ with

$$ad_H(X) = \alpha(H) X \quad \forall H \in \mathfrak{h}.$$

The root space $\mathfrak{g}_\alpha$ is the set of all such X (plus 0).

Let $\Phi = \Phi_{\mathfrak{g}}$ be the set of roots assoc. to $\mathfrak{g}$

Roots seem, unlikely since X must be a simultaneous e-vector for all of $\mathfrak{h}$.

However,

Thm 53 (Root space decomposition): If $\mathfrak{g}$ is semisimple w/ Cartan subalg $\mathfrak{h}$, then (as a v.s.),

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Pf sketch: Since $\mathfrak{h}$ is abelian, so is the set $\{ad_H \mid H \in \mathfrak{h}\}$ since by the Jacobi identity if $H, H' \in \mathfrak{h}$,

$$[H,[H',X]] + \underbrace{[X,[H,H']]}_{0} + \underbrace{[H',[X,H]]}_{-[H',[H,X]] \text{ by antisym.}} = 0$$

so $ad_H ad_{H'}(X) = ad_{H'} ad_H(X)$.

It is a fact (HW 4?) that a set of commuting diagonalizable operators is simultaneously diagonalizable. Choose a basis of $\mathfrak{g}$ (including a basis of $\mathfrak{h}$) such that the ad_H are simultaneously diagonalizable i.e. for all basis elts. X, $ad_H(X) = \alpha(H)X$ for some function $\alpha : \mathfrak{h} \to \mathbb{C}$.

Since $[\cdot,\cdot]$ is bilinear, α is a functional i.e. a root. Thus the basis consists of a basis for $\mathfrak{h}$ plus a basis for $\mathfrak{g}_\alpha$ for all $\alpha \in \Phi$. $\square$

Remark: We can think of $\mathfrak{h} =: \mathfrak{g}_0$ as the "root space" for the "root" $\alpha = 0$ since $\mathrm{ad}_H(X) = 0 \ \forall H, X \in \mathfrak{h}$.

Important properties:

- If $\alpha, \beta \in \Phi \sqcup \{0\}$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

  In particular, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$.

  Pf: Let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $H \in \mathfrak{h}$.

  Using the Jacobi identity,

$$
\begin{aligned}
[H,[X,Y]] &= [[H,X],Y] + [X,[H,Y]] \\
&= [\alpha(H)X, Y] + [X, \beta(H)Y] \\
&= (\alpha(H) + \beta(H))[X,Y]. \qquad \square
\end{aligned}
$$

- $\Phi_{\mathfrak{g}}$ is a root system, in the sense of Def. 45 (Hall Thm 6.34)

- All root spaces are one-dimensional (Hall Thm. 6.20)

  Furthermore, if $\alpha \in \Phi^+$, there exist nonzero elements

  $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{h}$ s.t.

  $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$, $[e_\alpha, f_\alpha] = h_\alpha$

  This is a copy of $\mathfrak{sl}_2$!

Example:

$$\mathfrak{g} = \mathfrak{sl}_n = \{\text{traceless } n \times n \text{ matrices}\}$$

Cartan subalg.:

$$\mathfrak{h} = \left\{ \text{diag. matrices } \underbrace{\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}}_{H} \;\middle|\; z_1 + \cdots + z_n = 0 \right\}$$

$$\mathfrak{g} = \mathfrak{h} \oplus \operatorname{span}\left\{ E_{ij} = \begin{array}{cc} \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} & i \\ j & \end{array} \;\middle|\; i \neq j \right\}$$

$$[H, E_{ij}] = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} - \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}$$

$$= z_i E_{ij} - z_j E_{ij}$$

$$= (z_i - z_j) E_{ij}$$

$$= \alpha_{ij}(H) E_{ij} \quad \text{where } \alpha_{ij}\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = z_i - z_j$$

The α_{ij} are exactly the A_{n-1} roots

$$\{ e_i - e_j \mid 1 \le i, j \le n,\ i \neq j \},$$

interpreting e_i as the functional $e_i\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = z_i$

potentially confusing notation

If $i<j$, $\alpha=\alpha_{ij}$, we have

$$e_\alpha = E_{ij}\ ,\quad f_\alpha = E_{ji}\ ,\quad h_\alpha = \begin{array}{cc} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} & \begin{array}{c} i \\ j \end{array} \\ \begin{array}{cc} i & \quad j \end{array} & \end{array}$$

If $n=3$, we have

$$\Delta$$

$$\Phi = \{ \underbrace{\overbrace{e_1-e_2}^{\alpha_1}, \overbrace{e_1-e_3}^{\alpha_1+\alpha_2}, \overbrace{e_2-e_3}^{\alpha_2}}_{\Phi^+}, e_2-e_1, e_3-e_1, e_3-e_2 \}$$

Root spaces:

$$\begin{bmatrix} \mathfrak{h} & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} \\ \mathfrak{g}_{-\alpha_1} & \mathfrak{h} & \mathfrak{g}_{\alpha_2} \\ \mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & \mathfrak{h} \end{bmatrix}$$

$$e_{\alpha_1} = \begin{bmatrix} & 1 & \\ & & \\ & & \end{bmatrix} \quad f_{\alpha_1} = \begin{bmatrix} & & \\ 1 & & \\ & & \end{bmatrix} \quad h_{\alpha_1} = \begin{bmatrix} 1 & & \\ & -1 & \\ & & \end{bmatrix}$$

$$e_{\alpha_2} = \begin{bmatrix} & & \\ & & 1 \\ & & \end{bmatrix} \quad f_{\alpha_2} = \begin{bmatrix} & & \\ & & \\ & 1 & \end{bmatrix} \quad h_{\alpha_2} = \begin{bmatrix} & & \\ & 1 & \\ & & -1 \end{bmatrix}$$