Recall:

Root system $\Phi \subseteq E$:

a) $\operatorname{span} \Phi = E$

b) If $\alpha \in \Phi$, then Φ contains $-\alpha$, but no other multiples of α (including 0).

c) For all $\alpha \in \Phi$, s_α preserves Φ

d) For all $\alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$

Rank is $\dim E$ Weyl group is $W = \{ s_\alpha \mid \alpha \in \Phi \}$

Every root system has a choice of simple roots: $\Delta \subseteq \Phi$ s.t

a) Δ is a basis of E

b) If $\beta \in \Phi$, then $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$,

$k_\alpha \in \mathbb{N}$ for all α

or

$k_\alpha \in -\mathbb{N}$ for all α

Given Δ, we can write $\Phi = \Phi^+ \sqcup \Phi^-$ where

$$\Phi^+ = \{\beta \in \Phi \mid \beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \text{ w/ } k_\alpha \geq 0\} \quad \text{positive roots}$$

$$\Phi^- = \{\beta \in \Phi \mid \beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \text{ w/ } k_\alpha \leq 0\} \quad \text{negative roots}$$

The <u>height</u> of $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ is $\sum_{\alpha \in \Delta} k_\alpha$.

Ex: $S = \{\alpha, \beta\}$ for all the rank 2 examples above

Properties (See Hall or Humphreys for proofs)

- $(\alpha, \beta) < 0$ for all $\alpha, \beta \in \Delta$
- Every root system has a unique <u>highest root</u> (w.r.t. a choice of simple roots)
- Given Δ, there exists a hyperplane $H \subseteq E$ s.t. Φ^+ and Φ^- lie on opposite sides of H.
- Let $\alpha, \beta \in \Phi$
    - If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root
    - If $(\alpha, \beta) > 0$, then $\alpha - \beta$ and $\beta - \alpha$ are roots

- Every $\beta \in \Phi$ can be written $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, $\alpha_i \in \Delta$ s.t. each partial sum $\alpha_1 + \cdots + \alpha_j$ is a root.

- W is generated by the set of <u>simple reflections</u>

$$S = \{ s_\alpha \mid \alpha \in \Delta \}.$$

- Let $\alpha \in \Delta$. Then s_α permutes $\Delta \setminus \{\alpha\}$.

- A <u>word</u> for $w \in W$ is a product

$$w = s_{i_1} \cdots s_{i_k}, \quad (s_{i_j} \in S)$$

The word is <u>reduced</u> if k is minimal (for w).

The <u>length</u> $\ell(w)$ of w is this k for a reduced word.

With this, we have:

$$\ell(w) = |\{ \alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^+ \}|$$

Classification of root systems

Let Φ be a root system, and fix an ordering $\alpha_1, \dots, \alpha_n$ of its simple roots. The <u>Cartan matrix</u> of Φ is the matrix

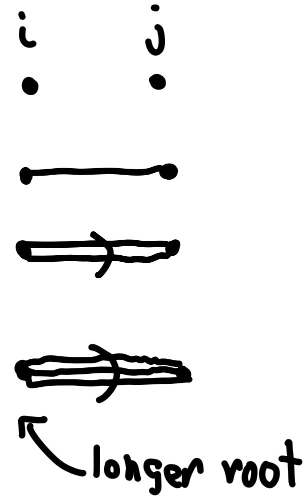
$$C = C_\Phi := \left(\langle \alpha_i, \alpha_j \rangle\right)_{ij}$$

e.g.

$$\underset{A_1 \times A_1}{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}} \quad \underset{A_2}{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}} \quad \underset{B_2/C_2}{\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}} \quad \underset{G_2}{\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}}$$

The <u>Dynkin diagram</u> associated to Φ (or C_Φ or W) is the graph with n vertices with the following connectivity

$C_{ij}C_{ji}$	m_{ij} ($= m_{\alpha_i,\alpha_j}$)	Angle	Length ratio	Dynkin diagram
0	2	$90°$	?	
1	3	$120°$	1	
2	4	$135°$	$\sqrt{2}$	
3	6	$150°$	$\sqrt{3}$	



A root system is Φ <u>irreducible</u> if it can't be written $\Phi = \Phi_1 \sqcup \Phi_2$ with $\Phi_2 \subseteq \Phi_1^\perp$

i.e. irreducible $\Leftrightarrow$ conn. Dynkin diag.

Two root systems are <u>equivalent</u> if some rotation/scaling sends one to the other

i.e. if they have the same Cartan matrix/Dynkin diagram

Thm 28 (Classification of root systems):

(see Hall/Humphreys for proof)

Up to equivalence, the irreducible root systems are:

Classical types:

A_n ($n \geq 1$)

$$\Phi = \{e_i - e_j \mid 1 \leq i,j \leq n+1,\ i \neq j\}$$

$$E = \operatorname{span} \Phi = \{v \in \mathbb{R}^{n+1} \mid \text{sum of coeffs.} = 0\}$$

$$\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n\}$$

$$C_\Phi = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

Dynkin diag.

$W \cong S_{n+1}$ (see HW3 #5a)

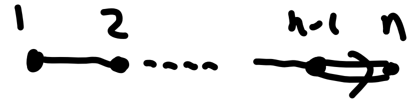
B_n $(n \geq 2)$

$$\Phi = \{ \pm e_i \pm e_j, \pm e_i \mid i \neq j \leq n \}$$

$$\Delta = \{ e_i - e_{i+1} \mid 1 \leq i \leq n-1 \} \cup \{ e_n \}$$

$$C_\Phi = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -2 & 2 \end{bmatrix}$$

Dynkin diag.



$W \cong \{ \text{signed permutations of } 1, \ldots, n \}$

C_n $(n \geq 2)$

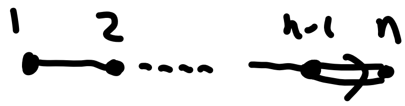
$((C_2 \cong B_2))$

$$\Phi = \{ \pm e_i \pm e_j, \pm 2e_i \mid i \neq j \leq n \}$$

$$\Delta = \{ e_i - e_{i+1} \mid 1 \leq i \leq n-1 \} \cup \{ 2e_n \}$$

$$C_\Phi = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -2 \\ & & & -1 & 2 \end{bmatrix}$$

Dynkin diag.



$W \cong \{ \text{signed permutations of } 1, \ldots, n \}$

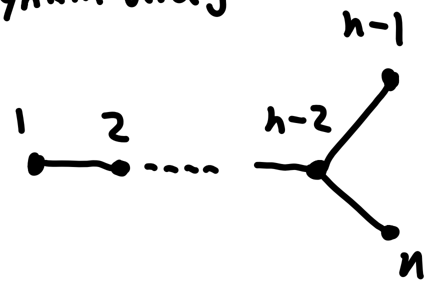
D_n $(n \geq 4)$

$\Phi = \{\pm e_i \pm e_j \mid i \neq j \leq n\}$

$\Delta = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\} \cup \{e_{n-1} + e_n\}$

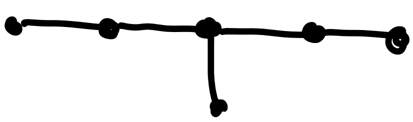
$$C_\Phi = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & & & & & \\ & & \ddots & -1 & & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{bmatrix}$$

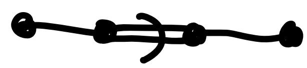
Dynkin diag.

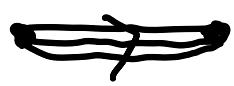


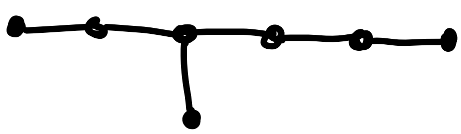
$W \cong \left\{\begin{array}{l}\text{signed permutations of } 1, \ldots, n \\ \text{w/ even \# signs flipped}\end{array}\right\}$

Exceptional types:

E_6 

F_4 

G_2 

E_7 

E_8