Announcements

Office hours (Harker 204C):
Monday 11:00am-11:50am
Wednesday 2:00pm-2:50pm
Friday 11:00am-11:50am

Representations of finite groups

Today: definitions, basic examples [Fulton-Harris §1.1]
Schur's Lemma [Fulton-Harris §1.2]

Def 1:

a) A representation (ρ, V) of a gp. G is a homom.

$$\rho : G \to GL(V)$$

Equivalently, (ρ, V) is a linear (left) action of G on V, and V is a v.s. that is also a (left) G-module

We will often use just ρ or V to express (ρ, V).

b) A subrepn. of (ρ, V) is a subspace $W \leq V$ that is G-invariant: $GW \subseteq W$. $\{0\}$ and V are always subrepns; if there are no others, V is called <u>irreducible</u>.

c) If V and W are G-repns., a <u>G-equivariant map</u> or <u>G-module homom.</u> is a linear map $\varphi : V \to W$ such that the diagrams

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

commute. If φ is a bijection it is an <u>isomorphism</u>. Let $\mathrm{Hom}_G(V, W)$ be the v.s. of G-equiv. maps $V \to W$ and $\mathrm{End}_G(V)$ be the ring $\mathrm{Hom}_G(V, V)$.

d) The <u>trivial repn</u> is (ρ_{triv}, V_{triv}) where $V_{triv} = F$ (field) and $\rho_{triv}(g) = [1]$ $\forall g \in G$.

Suppose V and W are G-repns. The following are G-repns:

- Direct sum:

$$V \oplus W \quad \text{via} \quad g \cdot (v+w) = g \cdot v + g \cdot w$$

- Tensor product:

$$V \otimes W \quad \text{via} \quad g \cdot (v \otimes w) = gv \otimes gw$$

- Tensor power: $V^{\otimes n}$
- Symmetric power:

$$\mathrm{Sym}^n V = V^{\otimes n} / (v_i \otimes v_j - v_j \otimes v_i)$$

- Exterior power:

$$\Lambda^n V = V^{\otimes n} / (v_i \otimes v_j + v_j \otimes v_i)$$

- $\mathrm{Hom}(V, W)$ via.

$$(g\varphi)(v) := g\varphi(g^{-1}v) \qquad \left(\begin{array}{c}\text{trivial on} \\ \mathrm{Hom}_G(V,W)\end{array}\right)$$

- Dual/contragredient:

$V^* = \mathrm{Hom}(V, V_{triv})$ via

$$(g\varphi)(v) := \varphi(g^{-1}v) \quad \text{or} \quad \langle \rho^*(g)v^*, v \rangle = \langle v^*, v \rangle$$

$$\text{or} \quad \rho^*(g) = \rho(g^{-1})^T$$

Class activity:

Let $\rho : GL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the identity repn

Compute $\rho^{\otimes 2}$, $\mathrm{Sym}^2 \rho$, $\Lambda^2 \rho$, and ρ^*

Lemma 2: Let $\varphi \in \mathrm{Hom}_G(V, W)$. Then $\ker \varphi$, $\mathrm{im}\, \varphi$, and $\mathrm{coker}\, \varphi$ are repns.

Pf: $v \in \ker \varphi \Rightarrow \varphi(g \cdot v) = g\varphi(v) = g \cdot 0 = 0$

$w \in \mathrm{im}\, \varphi \Rightarrow g \cdot w = g \underset{\exists v}{\varphi(v)} = \varphi(gv) \in \mathrm{im}\, \varphi$

This also implies that

$$g \cdot (\underbrace{x + \mathrm{im}\, \varphi}_{\in \mathrm{coker}\, \varphi}) := gx + \mathrm{im}\, \varphi \text{ is well-defined}$$

$\square$

In the case where $V \leq W$ and φ is the inclusion map, $\mathrm{coker}\, \varphi = W/V$ is called the <u>quotient repn</u>.

Def 3: A repn V is <u>completely reducible</u> if it is (isom to) a direct sum of irreps.:

multiplicities

$$V = V_1 \oplus \cdots \oplus V_m = \underbrace{c_1 V_1}_{\underbrace{V_1 \oplus \cdots \oplus V_1}_{c_1}} \oplus \cdots \oplus c_k V_k$$

mutually nonisom.

Recall Ex. 3 from last time was an example of a repn that was not completely reducible

Now for the next several weeks let us restrict to the case where G is a finite gp. and V is a f.d. complex v.s.

Prop 4 (Schur's Lemma): Let V, W be G-irreps. and $\varphi \in \mathrm{Hom}_G(V, W)$.

a) Either φ is an isom. or $\varphi = 0$.

b) If $V = W$, then φ is a scalar mult. of the identity

Pf:

a) $\ker\varphi$ and $\operatorname{im}\varphi$ are subrepns of V and W. Since V & W are irreps, we must have either

$$\underbrace{\ker\varphi = 0,\ \operatorname{im}\varphi = W}_{\text{isom.}} \quad \text{or} \quad \underbrace{\ker\varphi = V,\ \operatorname{im}\varphi = 0}_{\varphi = 0}.$$

b) Since $\mathbb{C}$ is alg. closed, φ must have an e-value λ. This means that $\ker(\varphi - \lambda I) \neq 0$, so by a), $\varphi - \lambda I$ is the zero map, and $\varphi = \lambda I$.

$\square$

Thm 5 (Maschke's Thm): Every finite dimensional complex repn. V of a finite gp. is completely reducible. Moreover, the decomposition

$$V = c_1 V_1 \oplus \cdots \oplus c_k V_k$$

is unique up to the order of the summands.