This week: Tondeur Lectures (June Huh)

Based on our computations last time, we have proved:

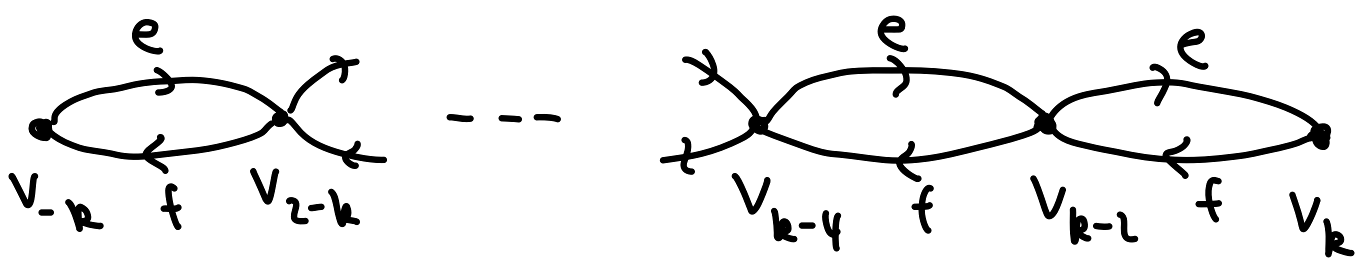
Thm 44: The irreducible f.d. representations of $\mathfrak{sl}_2(\mathbb{C})$ are parametrized by nonnegative integers k.
The irrep. $V^{(k)}$ has a unique vector v_k (up to scalar mult.) that satisfies $ev_k = 0$. Furthermore, $V^{(k)}$ has a basis of h-eigenvectors

$$V^{(k)} = \operatorname{span}\{v_k, v_{k-2}, \dots, v_{-k}\}$$

such that

$$\begin{aligned} h v_i &= i v_i \\ e v_{k-2l} &= l\, v_{k-2l+2} \\ f v_{k-2l} &= (k-l)\, v_{k-2l-2} \end{aligned}$$

$$\left[v_{k-2l} := (k-l)!\, f^l v \right]$$



Qualitative features:

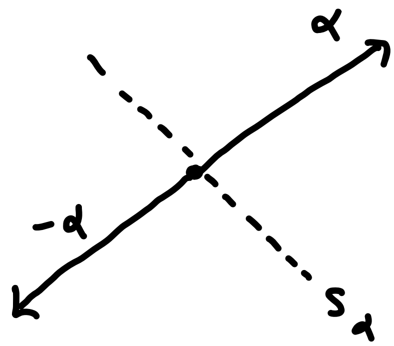
- Basis of h-eigenvectors
- e "raises" to a higher eigenspace, while f "lowers"
- Repn determined by the largest h-eigenvalue

Goal: generalize to a broader class of Lie algebras

<u>Root systems:</u>

Let E be a f.d. real v.s. For any nonzero $\alpha \in E$, let $s_\alpha \in GL(E)$ be the reflection across the hyperplane H_α passing through the origin and perpendicular to α.

s_α sends α to $-\alpha$



We have

$$s_\alpha(\beta) = \beta - \underbrace{2\frac{(\beta,\alpha)}{(\alpha,\alpha)}}_{:=\langle \beta,\alpha\rangle}\alpha$$

std. inner product on E

Def 45: A <u>root system</u> Φ is a finite subset of E (whose elements are called <u>roots</u>) which satisfies:

a) span $\Phi = E$

b) If $\alpha \in \Phi$, then Φ contains $-\alpha$, but no other multiples of α (including 0).

c) For all $\alpha \in \Phi$, s_α preserves Φ

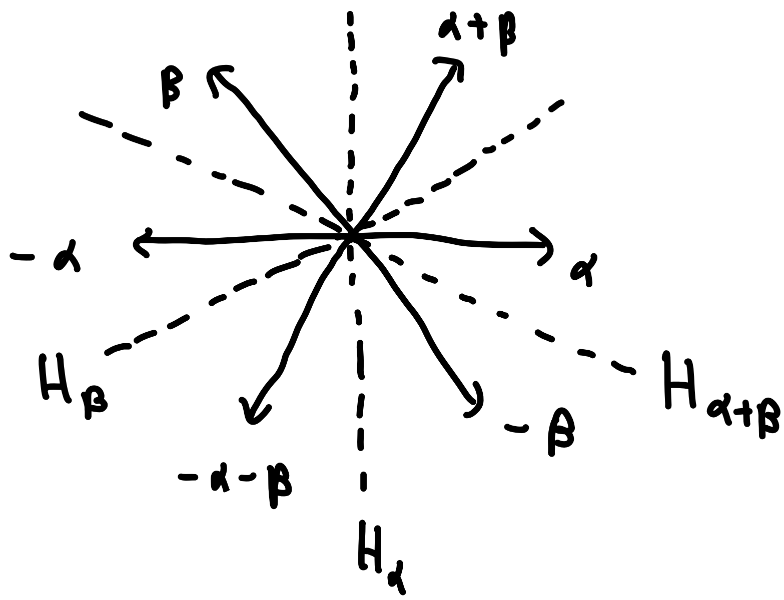
d) For all $\alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$

(• projection of β onto $\mathrm{span}(\alpha)$ is half-int. mult of α
• $s_\alpha(\beta) = \beta$ minus an integer mult. of α)

The <u>rank</u> of Φ is $\dim E$

The <u>Weyl group</u> of Φ is the subgp. $W \le GL(E)$ generated by the s_α.

Ex: A_2



$|\alpha| = |\beta|$
angle btwn α, β is $120°$

$$\langle \alpha, \alpha \rangle = 2 = \langle \beta, \beta \rangle$$

$$\langle \beta, \alpha \rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = -1 = \langle \alpha, \beta \rangle$$

$$W = \langle s_\alpha, s_\beta, s_{\alpha+\beta} \rangle$$

Class activity: determine what these reflections do to the roots and what relations they satisfy

What angles θ can we have btwn roots?

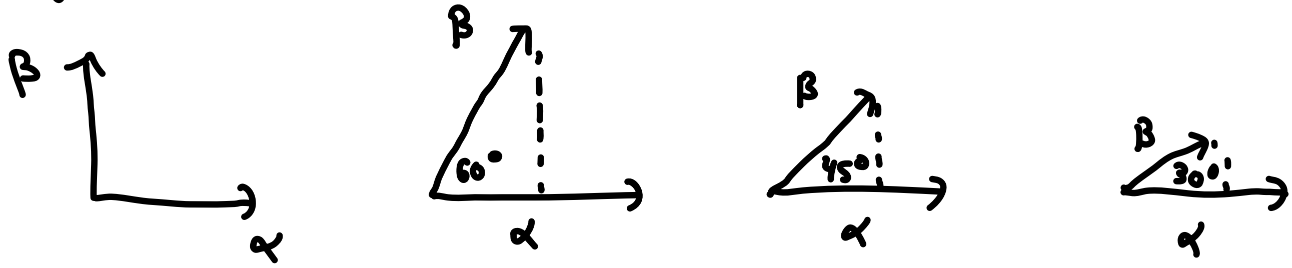
$$\underbrace{\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle}_{\in \mathbb{Z}} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \cdot 2\frac{(\alpha,\beta)}{(\beta,\beta)}$$

$$= 4\frac{(\alpha,\beta)^2}{\|\alpha\|^2\,\|\beta\|^2}$$

$$= (2\cos\theta)^2 \in [0,4]$$

$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle$	$\cos\theta$	θ	$\frac{\|\alpha\|}{\|\beta\|}$ $(\|\alpha\| \geq \|\beta\|)$
0	0	90°	any
1	$\pm\frac{1}{2}$	60°, 120°	1
2	$\pm\frac{\sqrt{2}}{2}$	45°, 135°	$\sqrt{2}$
3	$\pm\frac{\sqrt{3}}{2}$	30°, 150°	$\sqrt{3}$
4	± 1	0°, 180°	1 $(\beta = -\alpha)$

Length ratio:

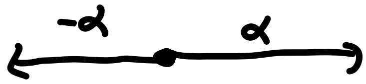


any 1 $\sqrt{2}$ $\sqrt{3}$

We have classified the rank ≤ 2 root systems:

Rank 1:

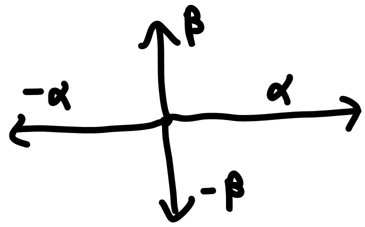
A_1



$$s_\alpha^2 = 1$$

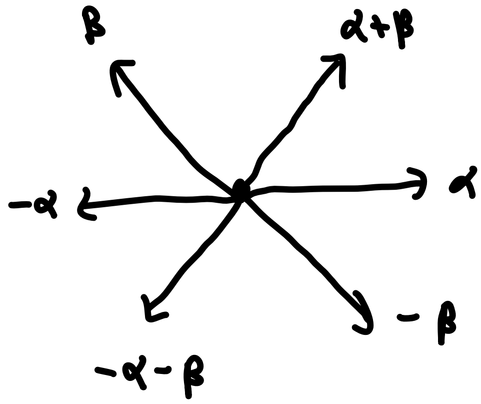
Rank 2:

$A_1 \times A_1$



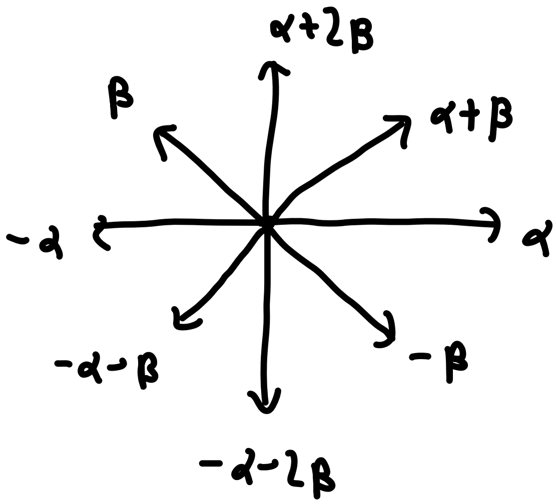
$$s_\alpha^2 = s_\beta^2 = 1$$
$$s_\alpha s_\beta = s_\beta s_\alpha$$

A_2



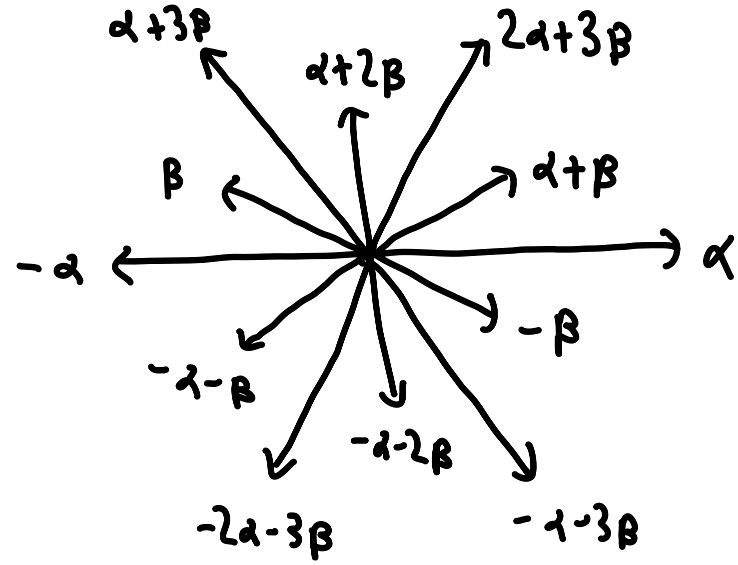
$$s_\alpha^2 = s_\beta^2 = 1$$
$$s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$$

B_2 / C_2



$$s_\alpha^2 = s_\beta^2 = 1$$
$$s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha s_\beta s_\alpha$$

G_2



$$s_\alpha^2 = s_\beta^2 = 1$$

$$s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha$$

For all these root systems,

$$W = \langle s_\alpha, s_\beta, s_\alpha^2 = s_\beta^2 = 1, \underbrace{s_\alpha s_\beta \cdots}_{m_{\alpha,\beta} \text{ factors}} = s_\beta s_\alpha \cdots \rangle$$

where $m_{\alpha,\beta} = \dfrac{180°}{180° - \theta}$ ← angle btwn. α and β

Def 46: A <u>base</u> is a subset $\Delta \subseteq \Phi$ s.t

a) Δ is a basis of E

b) If $\beta \in \Phi$, then $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$,

where $k_\alpha \in \mathbb{Z}$ for all α and the k_α are either all nonnegative or all nonpositive.

Elements of Δ are called <u>simple roots</u>

Given Δ, we can write $\Phi = \Phi^+ \sqcup \Phi^-$ where

$$\Phi^+ = \{\beta \in \Phi \mid \beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \text{ w/ } k_\alpha \geq 0\} \quad \text{positive roots}$$

$$\Phi^- = \{\beta \in \Phi \mid \beta = \sum_{\alpha \in \Delta} k_\alpha \alpha \text{ w/ } k_\alpha \leq 0\} \quad \text{negative roots}$$

The <u>height</u> of $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ is $\sum_{\alpha \in \Delta} k_\alpha$.

Ex: $S = \{\alpha, \beta\}$ for all the rank 2 examples above

Thm 47: Every root system has a base, which is unique up to the action of the Weyl group.

Pf: See [Hall, Thm. 8.14, Thm. 8.20] or [Humphreys, p. 48-51] $\square$

Properties (See Hall or Humphreys for proofs)

- $(\alpha, \beta) < 0$ for all $\alpha, \beta \in \Delta$
- Every root system has a unique <u>highest root</u> (w.r.t. a choice of simple roots)
- Given Δ, there exists a hyperplane $H \subseteq E$ s.t. Φ^+ and Φ^- lie on opposite sides of H.

- Let $\alpha, \beta \in \Phi$
  - If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root
  - If $(\alpha, \beta) > 0$, then $\alpha - \beta$ and $\beta - \alpha$ are roots
- Every $\beta \in \Phi$ can be written $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, $\alpha_i \in \Delta$ s.t. each partial sum $\alpha_1 + \cdots + \alpha_j$ is a root.
- W is generated by the set of <u>simple reflections</u>

$$S = \{ s_\alpha \mid \alpha \in \Delta \}.$$

- Let $\alpha \in \Delta$. Then s_α permutes $\Delta \setminus \{\alpha\}$.
- A <u>word</u> for $w \in W$ is a product

$$w = s_{i_1} \cdots s_{i_k}, \quad (s_{i_j} \in S)$$

The word is <u>reduced</u> if k is minimal (for w).

The <u>length</u> $\ell(w)$ of w is this k for a reduced word.

With this, we have:

$$\ell(w) = \left| \{ \alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^+ \} \right|$$

Next: classification of root systems