Starting new unit: representations of (complex) Lie algebras and Lie groups

- Sources: [Bump], [Humphreys], [Fulton-Harris], perhaps others

- No Lie theory background is assumed; we will state classification results where necessary
- Aiming for greatest hits (highest-weight repns, complete reducibility)

- Symmetric group will show up again, in two distinct ways!

Def 43:

- The general linear <u>Lie algebra</u> $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C})$ the v.s. of $n \times n$ complex matrices $\mathrm{Mat}_n(\mathbb{C})$.
- The special linear Lie algebra is the subspace $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) \subseteq \mathfrak{gl}_n$ of matrices w/ trace 0.

- The <u>Lie bracket</u> for $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{sl}_n$ is the commutator $[A, B] := AB - BA$

- A Lie algebra repn is a linear map

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

such that

$$\rho([A,B]) = [\rho(A), \rho(B)]$$

[Class activity: discuss why we might want to define $[\cdot,\cdot]$, and not just matrix multiplication.]

Our first task: classify the repns of $\mathfrak{sl}_2$.

Basis:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[h,e] = 2e \qquad [h,f] = -2f \qquad [e,f] = h$$

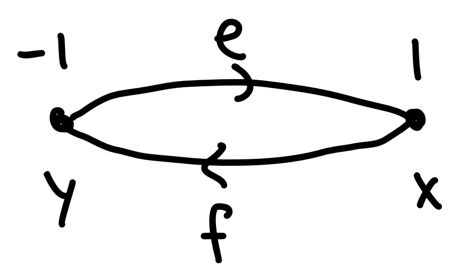
Ex:

a) Std. repn: $V = \mathbb{C}^2 = \mathrm{span}\left(x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

$hx = x \qquad hy = -y$

$ex = 0 \qquad ey = x$

$fx = y \qquad fy = 0$



b) Adjoint repn: $V = \mathfrak{sl}_2 \cong \mathbb{C}^3$

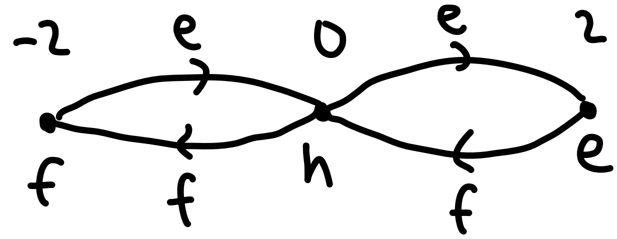
$$\mathrm{ad}: \mathfrak{g} \to \mathfrak{gl}(\underbrace{\mathfrak{g}}_{\text{v.s.}})$$

$$X \longmapsto \underbrace{\mathrm{ad}(X)}_{\in \mathfrak{gl}(\mathfrak{g})} \quad \text{where } \mathrm{ad}(X)(Y) := [X, Y]$$

$\mathrm{ad}(h)(h) = 0 \qquad \mathrm{ad}(h)(e) = 2e \qquad \mathrm{ad}(h)(f) = -2f$

$\mathrm{ad}(e)(h) = -2e \qquad \mathrm{ad}(e)(e) = 0 \qquad \mathrm{ad}(e)(f) = h$

$\mathrm{ad}(f)(h) = 2f \qquad \mathrm{ad}(f)(e) = -h \qquad \mathrm{ad}(f)(f) = 0$



Now let V be any (finite-dimensional) $\mathfrak{sl}_2$-irrep.

h has some e-vector $v \in V$ w/ e-value $\lambda \in \mathbb{C}$.

Then $hev = ehv + [h,e]v = e\lambda v + 2v$

$= (\lambda + 2)ev$

Similarly,

$hfv = (\lambda - 2)fv.$

So $e^i v$, $f^j v$ are linearly independent h-eigenvectors for all i,j. Finite dimensionality shows that eventually these become 0.

Let's take v to be an e-vector s.t. $ev=0$. Choose k maximal s.t. $f^k v \neq 0$. Then,

$$W = \operatorname{span}\{v, fv, f^2v, \dots, f^kv\} \leq V.$$

We claim that it's all of V. Clearly $fW \leq W$, and by iterating the Lie bracket,

$$hf^\ell v = (\lambda - 2\ell) f^\ell v \in W \qquad (hv = \lambda v)$$

Finally,

$$efv = fev + [e,f]v = 0 + hv = \lambda v$$

$$ef^2v = fefv + [e,f]fv = \lambda fv + hfv = (2\lambda - 2)fv$$

$$ef^3v = fef^2v + [e,f]f^2v = (2\lambda-2)f^2v + hf^2v = (3\lambda - 6)f^2v$$

$$\vdots$$

$$ef^\ell v = (\ell\lambda - \ell(\ell-1))f^{\ell-1}v$$

So $eW \leq W$, and since V is irred, $V = W$.

Last piece: what is λ?

$$0 = ef^{k+1} v = \big((k+1)\lambda - k(k+1)\big)\overbrace{f^k v}^{\neq 0}$$

So $(k+1)\lambda - k(k+1) = 0 \Rightarrow \lambda = k$

We have proved:

Thm 44: The irreducible f.d. representations of $\mathfrak{sl}_2(\mathbb{C})$ are parametrized by nonnegative integers k.
The irrep. $V^{(k)}$ has a unique vector v_k (up to scalar mult.) that satisfies $ev_k = 0$. Furthermore, $V^{(k)}$ has a basis of h-eigenvectors

$$V^{(k)} = \operatorname{span}\{v_k, v_{k-2}, \dots, v_{-k}\}$$

such that

$$hv_i = i v_i$$
$$ev_{k-2l} = l\, v_{k-2l+2}$$
$$fv_{k-2l} = (k-l)\, v_{k-2l-2}$$

$$\left[v_{k-2l} := (k-l)!\, f^l v \right]$$