Announcements

HW3 posted (due ???) ← 2 weeks from today is Spring Break

Midterm course feedback: (only 3 people responded)

Pacing good, resources good

Homework maybe a bit too long; want 2 full weeks to do it

Thm 40: In the decomp.

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda ,$$

we have $m_{\lambda,\mu} = K_{\lambda,\mu}$ (kostka number)

Pf: Step A: $M^\mu \cong \mathbb{C}[\mathcal{T}_{\lambda\mu}]$ (done)

Step B: For all $T \in \mathcal{T}_{\lambda\mu}$, let

$$\Theta_T : M^\lambda \to \mathbb{C}[\mathcal{T}_{\lambda\mu}] (\cong M^\mu)$$

$$\{t_\lambda\} \mapsto \sum_{S \in \{T\}} S$$

(extended by cyclicity)

Then the maps $\overline{\Theta_T} := \Theta_T|_{S^\lambda}$ form a basis of $\mathrm{Hom}_{S_n}(S^\lambda, S^\mu)$

Linear independence: done

For spanning, let $\varphi \in \mathrm{Hom}_{S_n}(S_\lambda, \underset{\wr\wr}{M^\mu})$.

Write $\varphi e_t = \sum_T c_T T \in \mathbb{C}[\mathcal{T}_{\lambda\mu}]$

$L_\varphi = \{ S \in \mathcal{T}^{ss}_{\lambda\mu} \mid [S] \trianglelefteq [T] \text{ for some } T \text{ appearing in } \varphi e_t \}$

Induction on $|L_\varphi|$:

- If $|L_\varphi| = 0$, then choose some T appearing in φe_t

  If T is not semistd.:

$$A\left\{\begin{array}[t]{c} a_i \\ \wedge \\ \vdots \\ a_p \end{array}\right. > \left.\begin{array}[b]{c} b_1 \\ \wedge \\ \vdots \\ \wedge \\ b_i \end{array}\right\} B$$

then applying the Garnir elt. $g_{A,B}$ to φe_t shows that $[T]$ is not maximal in φe_t.
i.e. if $|L_\varphi| = 0$, then $\varphi = 0$.

- Otherwise, choose T appearing in φe_T s.t. $[T]$ is maximal. Then T is semistd.

  Let

$$\varphi' = \varphi - c_T \overline{\Theta}_T.$$

$$\varphi' e_t = \sum_{T'} c_{T'} T' - c_T \sum_{S \in \{T\}} S$$

This subtracts off T and every S w/ $[S] = [T]$, so $L_{\varphi'} \subseteq L_\varphi$, and by induction, φ' and thus φ are in the span of the $\overline{\Theta}_{T'}$. □

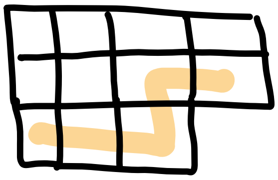
Characters of Specht modules (see HW3 for another rule)

χ^λ: character of S^λ

w_μ : permutation of cycle-type μ

A border-strip or rim hook ξ is a contiguous subdiagram of λ s.t. every cell is on the bottom-right border. Its size $|\xi|$ is the number of boxes. Its height $h(\xi)$ is the num. of rows it occupies, minus 1.

e.g.



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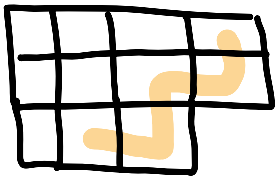

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Let $\mu' = (\mu_2, \mu_3, \dots, \mu_{\ell(\mu)})$

Thm 42 (Murnaghan-Nakayama rule): For all $\lambda, \mu \vdash n$,

$$\chi^\lambda(w_\mu) = \sum_\xi (-1)^{h(\xi)} \chi^{\lambda \setminus \xi}(w_{\mu'})$$

where the sum is over all border strips of λ of size μ_1.

Pf: much later in the course

Class activity: compute

$$\chi^{(4,4,3)}(w_{(5,4,2)}).$$

Next unit: representations of (complex) Lie algebras and Lie groups

- Sources: [Bump], [Humphreys], [Fulton-Harris], perhaps others

- No Lie theory background is assumed; we will state classification results where necessary
- Aiming for greatest hits (highest-weight repns, complete reducibility)

- Symmetric group will show up again, in two distinct ways!

Def 43:

- The general linear <u>Lie algebra</u> $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C})$ the v.s. of nxn complex matrices $\mathrm{Mat}_n(\mathbb{C})$.
- The special linear Lie algebra is the subspace $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) \subseteq \mathfrak{gl}_n$ of matrices w/ trace 0.

- The <u>Lie bracket</u> for $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{sl}_n$ is the commutator $[A, B] := AB - BA$

- A Lie algebra repn is a linear map

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$$

such that

$$\rho([A,B]) = [\rho(A), \rho(B)]$$

[Class activity: discuss why we might want to define [·,·], and not just matrix multiplication.]

Our first task: classify the repns of $\mathfrak{sl}_2$.

Basis:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[h,e] = 2e \qquad [h,f] = -2f \qquad [e,f] = h$$

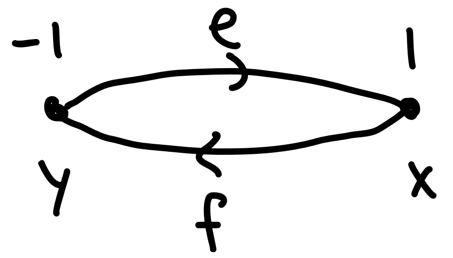
Ex:

a) Std. repn: $V = \mathbb{C}^2 = \text{span}\left(x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

$hx = x \qquad hy = -y$

$ex = 0 \qquad ey = x$

$fx = y \qquad fy = 0$



b) Adjoint repn: $V = \mathfrak{sl}_2 \cong \mathbb{C}^3$

$$\text{ad}: \mathfrak{g} \to \mathfrak{gl}(\underbrace{\mathfrak{g}}_{\text{v.s.}})$$

$$X \longmapsto \underbrace{\text{ad}(X)}_{\in \mathfrak{gl}(\mathfrak{g})} \quad \text{where} \quad \text{ad}(x)(Y) := [x, Y]$$

$\text{ad}(h)(h) = 0$ $\quad$ $\text{ad}(h)(e) = 2e$ $\quad$ $\text{ad}(h)(f) = -2f$

$\text{ad}(e)(h) = -2e$ $\quad$ $\text{ad}(e)(e) = 0$ $\quad$ $\text{ad}(e)(f) = h$

$\text{ad}(f)(h) = 2f$ $\quad$ $\text{ad}(f)(e) = -h$ $\quad$ $\text{ad}(f)(f) = 0$