Today: Decomposition of M^μ (cont.) [Sagan §2.9-11]
[James §13-14]

We are working towards:

Thm 40: In the decomp.

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda \quad ,$$

we have $m_{\lambda,\mu} = K_{\lambda,\mu}$ ← kostka number

Recall:

$\mathcal{T}_{\lambda\mu} = \{$tableaux of shape λ and content $\mu\}$

$\mathcal{T}_{\lambda\mu}^{ss} = \{T \in \mathcal{T}_{\lambda\mu} \mid T$ is semistd$\}$

$\Theta : M^\mu \to \mathbb{C}[\mathcal{T}_{\lambda\mu}]$

$\{s\} \mapsto T$ where $T(i) =$ the row in which i appears in $\{s\}$.

$S_n \curvearrowright \mathcal{T}_{\lambda\mu}$ via $\sigma T(i) := T(\sigma^{-1} i)$

The pf of Thm 40 has two steps:

Step A: $M^\mu \cong \mathbb{C}[\mathcal{T}_{\lambda\mu}]$

Step B: For all $T \in \mathcal{T}_{\lambda\mu}$, let

$$\theta_T : M^\lambda \to \mathbb{C}[\mathcal{T}_{\lambda\mu}] \, (\cong M^\mu)$$

$$\{t_\lambda\} \mapsto \sum_{S \in \{T\}} S$$

(extended by cyclicity)

Then the maps $\theta_T|_{S^\lambda}$ form a basis of $\mathrm{Hom}_{S_n}(S^\lambda, S^\mu)$

Class activity: compute $\theta_T\{t_\lambda\}$ and $\theta_T \left\{ \begin{array}{lll} 2 & 4 & 3 \\ 1 & 5 & \end{array} \right\}$

where $T = \begin{array}{lll} 2 & 1 & 1 \\ 3 & 2 & \end{array}$

$$\theta_T\{t_\lambda\} = \begin{array}{lll} 2 & 1 & 1 \\ 3 & 2 & \end{array} + \cdots$$

$$\theta_T \left\{ \begin{array}{lll} 2 & 4 & 3 \\ 1 & 5 & \end{array} \right\} = \theta_T (1,2,4) t_\lambda = (1,2,4)\, \theta_T\, t_\lambda = \begin{array}{lll} 3 & 1 & 2 \\ 1 & 2 & \end{array} + \cdots$$

Pf of Step A: θ is bijective by inspection.

It is S_n-equivariant since if $\theta\{S\} = T$, then

$$\sigma\theta\{S\} = \sigma T(i) = T(\sigma^{-1} i) = \text{row num. of } \sigma^{-1} i \text{ in } \{S\}$$

$$= \text{row num. of } i \text{ in } \sigma\{S\}$$

$$= \theta(\sigma\{S\})(i). \qquad \square$$

Lemma 41: Dominance lemma for column tabloids w/ repetition:

a) Let $k < l$. If k appears in a column to the left of l in T, then $[T] \underset{\neq}{\trianglerighteq} [S]$, where S is the tableau obtained by swapping the k and l in T.

b) If T semistd. and $S \in \{T\}$, then $[T] \trianglerighteq [S]$.

Pf: Similar to past dominance lemmata. $\square$

Pf of Step B: Has similarities to pf of Thm 33, so we sketch

Let $\overline{\Theta_T} = \Theta_T|_{S^\lambda}$. First we claim that the elts.

$\{\overline{\Theta_T} \mid T \in \mathcal{T}^{ss}_{\lambda\mu}\}$ is linearly indep.

Apply them to e_t:

stabilizes cols.

$$\overline{\Theta_T} e_t = \Theta_T k_t\{t\} = k_t \Theta_T\{t\} = k_t \sum_{S' \in \{T\}} S'.$$

If S appears in this sum, then by Lemma 41b, $[T] \trianglerighteq [S]$. Thus by triangularity, the $\overline{\Theta_T} e_t$ and hence the $\overline{\Theta_T}$ are linearly indep.

For spanning, let $\varphi \in \mathrm{Hom}_{S_n}(S_\lambda, \underset{\wr\wr}{M^\mu})$.

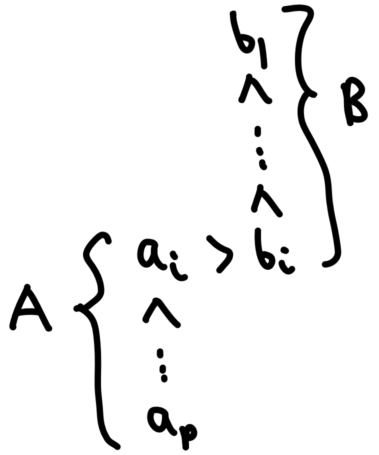
Write $\varphi e_t = \sum_T c_T T \in \mathbb{C}[\mathcal{T}_{\lambda\mu}]$

$L_\varphi = \{ S \in \mathcal{T}^{ss}_{\lambda\mu} \mid [S] \trianglelefteq [T] \text{ for some } T \text{ appearing in } \varphi e_t \}$

Induction on $|L_\varphi|$:

- If $|L_\varphi| = 0$, then choose some T appearing in φe_t

  If T is not semistd.:



then applying the Garnir elt. $g_{A,B}$ to φe_t shows that $[T]$ is not maximal in φe_t.
i.e. if $|L_\varphi| = 0$, then $\varphi = 0$.

- Otherwise, choose T appearing in φe_T s.t. $[T]$ is maximal. Then T is semistd.

  Let
$$\varphi' = \varphi - c_T \overline{\Theta}_T .$$
$$\varphi' e_t = \sum_{T'} c_{T'} T' - c_T \sum_{S \in \{T\}} S$$

  This subtracts off T and every S w/ $[S] = [T]$, so $L_{\varphi'} \subseteq L_\varphi$, and by induction, φ' and thus φ are in the span of the $\overline{\Theta}_{T'}$. □

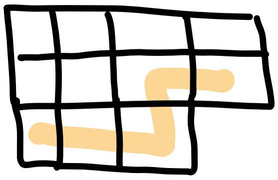
Characters of Specht modules (see HW3 for another rule)

χ^λ: character of S^λ

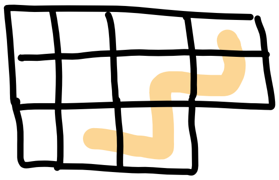
w_μ : permutation of cycle-type μ

A **border-strip** or **rim hook** ξ is a contiguous subdiagram of λ s.t. every cell is on the bottom-right border. Its **size** $|\xi|$ is the number of boxes. Its height $h(\xi)$ is the num. of rows it occupies, minus 1.

e.g.



size 5
height



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Let $\mu' = (\mu_2, \mu_3, \dots, \mu_{\ell(\mu)})$

Thm 42 (Murnaghan–Nakayama rule): For all $\lambda, \mu \vdash n$,

$$\chi^\lambda(w_\mu) = \sum_\xi (-1)^{h(\xi)} \chi^{\lambda \setminus \xi}(w_{\mu'})$$

where the sum is over all border strips of λ of size μ_1.

Pf: much later in the course

(If time): class activity: compute

$$\chi^{(4,4,3)}(w_{(5,4,2)}).$$

Next time:
$\mathfrak{sl}_2(\mathbb{C})$-repns.