Announcements

HW2 sol'ns posted

Grading

We are proving:

Thm 39 (Branching rule for S_n):

a) $\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\mu \in \lambda^-} S^\mu$

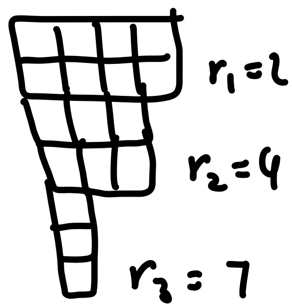
mult. free!

b) $\mathrm{Ind}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\mu \in \lambda^+} S^\mu$

By Frobenius reciprocity, these are equivalent.

Pf: Let $r_1 < r_2 < \cdots < r_k$ denote the corners of λ. Let λ^i be λ w/out the corner in row r_i.

e.g.



Let

$V^{(i)} = \mathrm{span}\{e_T \in E^\lambda \mid n \text{ appears in first } r_i \text{ rows of } T\}$

We have $\phi \subseteq V^{(1)} \subseteq \cdots \subseteq V^{(k)} = S^\lambda$.

We claim that the $V^{(i)}$ are subreps, and that $V^{(i)}/V^{(i-1)} \cong S^{\lambda^i}$.

Then the result follows since by Maschke's Thm,

$$S^\lambda \cong V^{(1)} \oplus V^{(2)}/V^{(1)} \oplus \cdots \oplus V^{(k)}/V^{(k-1)}.$$

Let

$$\theta_i : M^\lambda \to M^{\lambda^i}$$

$$\{T\} \mapsto \begin{cases} \{T \setminus n\}, & \text{if } n \text{ is in row } r_i \\ 0, & \text{otherwise} \end{cases}$$

θ_i is S_{n-1}-equivariant and for T: std.,

$$\theta_i(e_T) = \begin{cases} e_{T\setminus n}, & \text{if } n \text{ is in row } r_i \\ 0, & \text{if } n \text{ is in a higher row} \end{cases}$$

since if $\{S\}$ appears in e_T, standardness of T means that n appears in a weakly higher row of $\{S\}$.

Thus we have $\theta_i V^{(i)} = S^{\lambda^i}$ and $V^{(i-1)} \subseteq \ker \theta$.

Note that

$$\dim S^\lambda = \sum_{\mu \in \lambda^-} \dim S^\mu \qquad \left(f^\lambda = \sum_{\mu \in \lambda^-} f^\mu\right)$$

Since removing n from a std. tableau for λ gives a std. tableau for some S^μ.

In the chain

$$\{0\} \subseteq V^{(1)} \cap \ker\theta_1 \subseteq V^{(1)} \subseteq V^{(2)} \cap \ker\theta_2 \subseteq V^{(2)} \subseteq \cdots \subseteq V^{(k)} = S^\lambda,$$

we have

$$\dim \frac{V^{(i)}}{V^{(i)} \cap \ker\theta} = \dim \theta V^{(i)} = f^{\lambda^i},$$

so

$$\sum_i \dim \frac{V^{(i)}}{V^{(i)} \cap \ker\theta} = \sum_i f^{\lambda^i} = f^\lambda = \dim S^\lambda$$

and so we must have $V^{(i)} \cap \ker\theta = V^{(i-1)}$ $\forall i$.

Hence, $V^{(i)}/V^{(i-1)} \cong S^{\lambda^i}$.

$\square$

Recall (Thm 32):

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda, \qquad m_{\mu,\mu} = 1$$

We'll work w/ tableau w/ (potentially) repeated entries

The <u>content</u> of T is the composition $a = (a_1, \ldots, a_m)$ where a_i is the number of i's in T.

Row/col. tabloids defined similarly to the no-repeats case

e.g. $\begin{matrix} 4 & 1 & 4 \\ 1 & 3 & \end{matrix}$ shape: $\lambda = (3,2)$
content: $\mu = (2,0,1,2)$

Recall that T is <u>semistandard</u> if entries weakly increase along rows and strictly increase down columns.

Slightly reinterpreting the dominance lemma (Lemma 21), if T is semistandard, then shape$(T) \trianglerighteq$ content(T).

The <u>Kostka number</u> $K_{\lambda,\mu}$ is the number of SSYT of shape λ and content μ.

We will prove:

Thm 40: $m_{\lambda,\mu} = K_{\lambda,\mu}$ (λ,μ: partitions)

Class activity: Decompose $M^{(2,2,1)}$ into irreps.

Example: $K_{\lambda,1^n} = f^\lambda$, so

$$M^{(1^n)} = \bigoplus_{\lambda \vdash n} f^\lambda S^\lambda$$ (recall that $M_{(1^n)}$ is the reg. repn)

Let

$$\mathcal{T}_{\lambda\mu} = \{\text{tableaux of shape } \lambda \text{ and content } \mu\}$$

$$\mathcal{T}_{\lambda\mu}^{ss} = \{T \in \mathcal{T}_{\lambda\mu} \mid T \text{ is semistd}\}$$

For any shape λ, fix the tableau $t := t_\lambda$

$$t = \begin{matrix} 1\ 2 \cdots \lambda_1 \\ \lambda_1{+}1 \cdots \lambda_1{+}\lambda_2 \\ \cdots \\ \cdots |\lambda| \end{matrix}$$

shape λ
increasing $1, \ldots, |\lambda|$
in English reading order

For any λ-tableau T, let

$T(i)$ = entry of T in the spot where t_λ has i.

e.g. $$T = \begin{matrix} T(1) & T(2) & T(3) \\ T(4) & T(5) \end{matrix}$$

Let $\Theta : M^\mu \to \mathbb{C}[\mathcal{T}_{\lambda\mu}]$

$\{s\} \mapsto T$ where $T(i)$ = the row in which i appears in $\{s\}$.

Let $S_n \curvearrowright \mathcal{T}_{\lambda\mu}$ via

$$\sigma T(i) := T(\sigma^{-1} i)$$

e.g. $$(124) \cdot \begin{matrix} 2 & 1 & 1 \\ 3 & 2 \end{matrix} = \begin{matrix} 3 & 2 & 1 \\ 1 & 2 \end{matrix}$$ since

$$(124) \cdot \begin{matrix} T(1) & T(2) & T(3) \\ T(4) & T(5) \end{matrix} = \begin{matrix} T(4) & T(1) & T(3) \\ T(2) & T(5) \end{matrix}$$

The pf of Thm 40 has two steps:

Step A: $M^\mu \cong \mathbb{C}[\mathcal{T}_{\lambda\mu}]$

Step B: For all $T \in \mathcal{T}_{\lambda\mu}$, let

$$\theta_T : M^\lambda \to \mathbb{C}[\mathcal{T}_{\lambda\mu}] \; (\cong M^\mu)$$

$$\{t_\lambda\} \mapsto \sum_{S \in \{T\}} S$$

(extended by cyclicity)

Then the maps $\theta_T|_{S^\lambda}$ form a basis of $\mathrm{Hom}_{S_n}(S^\lambda, S^\mu)$

Class activity: compute $\theta_T\{t_\lambda\}$ and $\theta_T\left\{\begin{matrix} 2 & 4 & 3 \\ 1 & 5 & \end{matrix}\right\}$

where $T = \begin{matrix} 2 & 1 & 1 \\ 3 & 2 & \end{matrix}$