Today: Basis of S^λ (cont.) [Sagan §2.5-2.6]

Heading towards: [James §7, 8]

Thm 33: The set

$$E^\lambda = \{e_T \mid T \text{ is a std. tableau of shape } \lambda\}$$

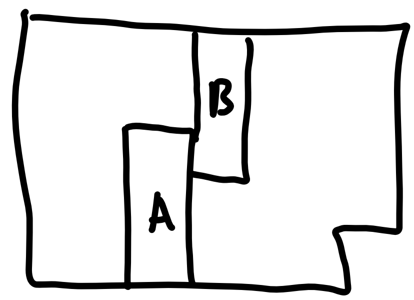
is a basis for S^λ.

Last time: E^λ is linearly indep.

Today: E^λ spans S^λ.

Recall: Garnir elts.

$$g_{A,B} := \sum_{i=1}^{k} (-1)^{\sigma_i} \sigma_i$$



We are proving:

Prop 37: $g_{A,B}\, e_T = 0$

Pf: Last time: $S^-_{A \cup B}\, e_T = 0$.

Now,
$$S^-_{A\cup B} = \sum_{w \in S_{A \cup B}} (-1)^w w = \sum_{\sigma \in S_{A\cup B}/S_A \times S_B} (-1)^\sigma \sigma \sum_{w \in S_A \times S_B} (-1)^w w$$

$$= g_{A,B}\,(S_A \times S_B)^-,$$

So $g_{A,B}(S_A \times S_B)^- e_T = 0$, and the result will follow if $(S_A \times S_B)^- e_T$ is a nonzero multiple of e_T.

Since $S_A \times S_B \subseteq C_T$, if $w \in S_A \times S_B$, then

$$(-1)^w w e_T = (-1)^w K_T \{T\} = K_T \{T\} e_T,$$

so $(S_A \times S_B)^- e_T = |S_A \times S_B| e_T$. $\square$

We'll use <u>column tabloids</u>

$$[T] := C_T T$$

e.g. $\left[\begin{matrix} 1 & 2 \\ 3 & \end{matrix}\right] = \left\{ \begin{matrix} 1 & 2 \\ 3 & \end{matrix}, \begin{matrix} 3 & 2 \\ 1 & \end{matrix} \right\} = \left|\begin{matrix} 1 \\ 3 \end{matrix}\right|\begin{matrix} 2 \\ \end{matrix}\right|$

Dominance order induced from the map $[T] \mapsto \{T'\}$ and similar facts hold (e.g. dominance lemma)

Pf of Thm 33:
We've already proved linear independence, just need to prove span $E^\lambda = S^\lambda$.

Induction of column tabloid dominance order.

Let $T_0 = \begin{matrix} 1 & a+1 & b+1 & \cdots \\ 2 & \vdots & \vdots & \\ 3 & \vdots & \vdots & \\ \vdots & b & & \\ a & & & \end{matrix}$ be the tableau numbered by columns. T_0 is std. and $[T_0] \trianglerighteq [T]$ $\forall T$.

Fix a tableau T. If $S \in T$, then $e_S = \pm e_T$, so we can assume T has increasing columns.

Assume $e_S \in \operatorname{span} E^\lambda$ whenever $[S] \trianglerighteq [T]$.

If T std., done. Otherwise, T has a descent along a row:

$$\begin{array}{ccccc}
 & a_1 & & b_1 & \\
 & \vdots & & \vdots & \\
 & & & \wedge & B \\
 & a_i & > & b_i & \\
A & \wedge & & \vdots & \\
 & \vdots & & b_q & \\
 & a_p & & &
\end{array}$$

By Prop 37, $g_{A,B}\, e_T = 0$, so

$$e_T = -\sum_{\substack{j \\ \sigma_j \neq 1}} (-1)^{\sigma_j} e_{\sigma_j T} \qquad (*)$$

Now, each σ_j is the product of permutations of A, permutations of B, and transpositions (a_i, b_j) w/ $a_i > b_j$.

By the dominance lemma for column tabloids, $[\sigma_j T] \triangleright T$, so by induction, $e_T \in \operatorname{span} E^\lambda$. $\square$
$\sigma_j \neq 1$

Let f_λ be the number of std. tableaux of shape λ.

Cor 38:

a) $\dim S^\lambda = f^\lambda$

b) $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$

c) With respect to the basis E^λ, the matrices for the repn S^λ have integer entries.
(Young's natural repn)

d) The character table for S_n has integer entries.

Pf: a) $f^\lambda = |E^\lambda|$.

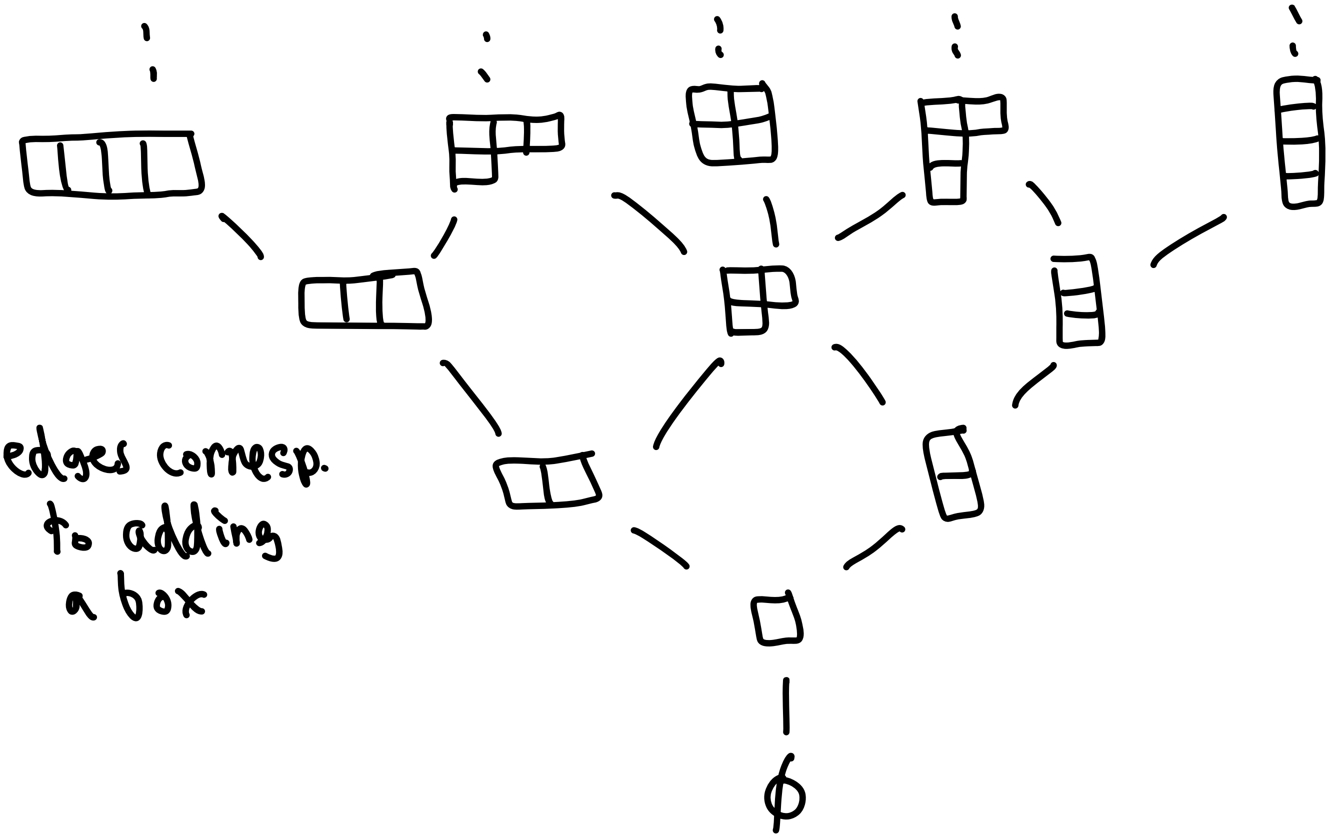
b) Apply Cor 14: $|G| = \sum_{V:\text{irrep}} (\dim V)^2$

c) $w e_T = e_{wT}$, and by (*) each polytabloid is an integer linear comb. of std. polytabloids.

d) The trace of an integer matrix is an integer. $\square$

Recall: containment order $\lambda \subseteq \mu$ if $\lambda_i \le \mu_i \ \forall i$

This poset is called Young's lattice:



edges corresp. to adding a box

Let $\lambda^+ = \left\{ \mu \text{ s.t. } \begin{smallmatrix} \mu \\ | \\ \lambda \end{smallmatrix} \right\}$ $\quad \lambda^- = \left\{ \mu \text{ s.t. } \begin{smallmatrix} \lambda \\ | \\ \mu \end{smallmatrix} \right\}$

Thm 39 (Branching rule for S_n):

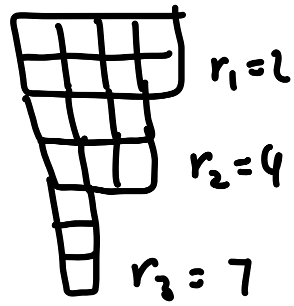
a) $\mathrm{Res}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\mu \in \lambda^-} S^\mu$

mult. free!

b) $\mathrm{Ind}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\mu \in \lambda^+} S^\mu$

By Frobenius reciprocity, these are equivalent.

e.g.



Pf: Let $r_1 < r_2 < \cdots < r_k$ denote the corners of λ. Let λ^i be λ w/out the corner in row r_i.

Let

$$V^{(i)} = \operatorname{span}\{e_T \in E^\lambda \mid n \text{ appears in first } r_i \text{ rows of } T\}$$

We have $\phi \subseteq V^{(1)} \subseteq \cdots \subseteq V^{(k)} = S^\lambda$.

We claim that the $V^{(i)}$ are subrepns, and that $V^{(i)}/V^{(i-1)} \cong S^{\lambda^i}$.

Then the result follows since by Maschke's Thm,

$$S^\lambda \cong V^{(1)} \oplus V^{(2)}/V^{(1)} \oplus \cdots \oplus V^{(k)}/V^{(k-1)}.$$

Let

$$\Theta_i : M^\lambda \to M^{\lambda^i}$$

$$\{T\} \mapsto \begin{cases} \{T \setminus n\}, & \text{if } n \text{ is in row } r_i \\ 0, & \text{otherwise} \end{cases}$$

Θ_i is S_{n-1}-equivariant and for T: std.,

$$\Theta_i(e_T) = \begin{cases} e_{T \setminus n}, & \text{if } n \text{ is in row } r_i \\ 0, & \text{if } n \text{ is in a higher row} \end{cases}$$

since if $\{s\}$ appears in e_T, standardness of T means that n appears in a weakly higher row of $\{s\}$.

Thus we have $\theta_i V^{(i)} = S^{\lambda^i}$ and $V^{(i-1)} \subseteq \ker\theta$.

Note that

$$\dim S^\lambda = \sum_{\mu\in\lambda^-} \dim S^\mu \qquad \left(f^\lambda = \sum_{\mu\in\lambda^-} f^\mu\right)$$

Since removing n from a std. tableau for λ gives a std. tableau for some S^μ.

In the chain

$$\{0\} \subseteq V^{(1)}\cap\ker\theta_1 \subseteq V^{(1)} \subseteq V^{(2)}\cap\ker\theta_2 \subseteq V^{(2)} \subseteq \cdots \subseteq V^{(k)} = S^\lambda,$$

we have

$$\dim \frac{V^{(i)}}{V^{(i)}\cap\ker\theta} = \dim \theta V^{(i)} = f^{\lambda^i},$$

so

$$\sum_i \dim \frac{V^{(i)}}{V^{(i)}\cap\ker\theta} = \sum_i f^{\lambda^i} = f^\lambda = \dim S^\lambda$$

and so we must have $V^{(i)}\cap\ker\theta = V^{(i-1)}$ $\forall i$.

Hence, $V^{(i)}/V^{(i-1)} \cong S^{\lambda^i}$.

$\square$