Today: Basis of S^λ [Sagan §2.5-2.6]

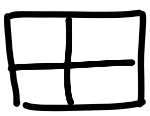
Heading towards: [James §7, 8]

Thm 33: The set

$$E^\lambda = \{e_T \mid T \text{ is a std. tableau of shape } \lambda\}$$

is a basis for S^λ.

Recall: $\{S\} \trianglerighteq \{T\}$ if $\lambda^i(S) \trianglerighteq \lambda^i(T)$ $\forall i$.

Class activity: draw the poset of tabloids of shape 

Lemma 34 (Dominance Lemma for tabloids): If $k<\ell$ and k appears in a lower row than ℓ in $\{T\}$, then

$$\{T\} \underset{\neq}{\trianglelefteq} (k,\ell)\{T\} =: \{S\}$$

Pf: For $i<k$ and $i \geq \ell$ we have $\lambda^i(S) = \lambda^i(T)$.

If $k \leq i < \ell$, let r and q be the rows of $\{T\}$ in which k and ℓ appear. Then

$\lambda^i(S) = \lambda^i(T)$ w/ qth part increased by 1

and rth part decreased by 1

Since $q<r$, $\lambda^i(S) \trianglerighteq \lambda^i(T)$. $\square$

Proposition 35: Let T be a std. tableau. If $\{S\}$ appears in e_T, then $\{T\} \trianglerighteq \{S\}$. Furthermore, E^λ is linearly independent.

Pf: If $\{S\}$ appears in T, then $S = \sigma T$ for some $\sigma \in C_T$ and some choice of representative $S \in \{S\}$.

Induction on <u>column inversions</u> in S:

Pairs $k<\ell$ $\quad \begin{array}{c} \ell \\ \vdots \\ k \end{array}$ same col.

By Dominance Lemma, $S \triangleleft (k,\ell)\{S\}$.

Since T is std., it has no col. inversions, and σ is a product of col.-inv.-reducing transpositions. Thus, the decomp. of the std. polytabloids into tabloids is triangular; hence E^λ is linear independent

$\square$

all tabloids, ordered ~ dominance

$$\begin{array}{rc}
 & \begin{array}{ccccc} \{T_1\} & \cdots & & \{T_k\} & \cdots \end{array} \\
\text{std. polytab. ordered} \sim \text{dominance} \left\{ \begin{array}{c} e_{T_1} \\ e_{T_2} \\ \vdots \\ e_{T_k} \end{array} \right. & \begin{bmatrix} 1 & & & & * \\ & 1 & & & \\ 0 & & \ddots & & \\ & & & 1 & \end{bmatrix}
\end{array}$$

Now for spanning:

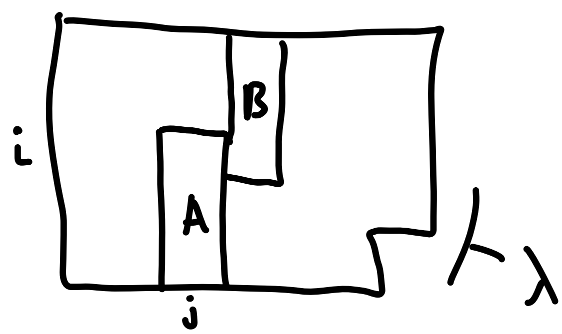
Def 36: a) Let A and B be disjoint subsets of $[n]$. Fix coset reps. $\sigma_1, \dots, \sigma_k$ for $S_{A \cup B} / S_A \times S_B$.

The corresp. <u>Garnir elt.</u> is

$$g_{A,B} := \sum_{i=1}^{k} (-1)^{\sigma_i} \sigma_i$$

b) If A and B are these entries of a tableau:

T :



adjacent cols, one row overlap, union has all rows then we choose $\sigma_1, \dots, \sigma_k$ s.t. in $\sigma_i T$ the entries in A and B are increasing down cols.

Class activity: Find the Garnir elt $g_{A,B}$ with

$$T = \begin{array}{lll} 1 & 2 & 3 \\ 5 & 4 & \\ 6 & & \end{array} \qquad A = \{5, 6\} \qquad B = \{2, 4\}$$

Ans: $g_{A,B} = () - (45) + (245) + (465) - (2465) + (25)(46)$

$$\begin{array}{lll} 1 & 2 & 3 \\ 5 & 4 & \\ 6 & & \end{array} \quad \begin{array}{lll} 1 & 2 & 3 \\ 4 & 5 & \\ 6 & & \end{array} \quad \begin{array}{lll} 1 & 4 & 3 \\ 2 & 5 & \\ 6 & & \end{array} \quad \begin{array}{lll} 1 & 2 & 3 \\ 4 & 6 & \\ 5 & & \end{array} \quad \begin{array}{lll} 1 & 4 & 3 \\ 2 & 6 & \\ 5 & & \end{array} \quad \begin{array}{lll} 1 & 5 & 3 \\ 2 & 6 & \\ 4 & & \end{array}$$

decreasing

increasing

Prop 37: Under this setup, $g_{A,B} e_T = 0$

If $H \subseteq S_n$, let $H^- := \sum_{w \in H} (-1)^w w$

Pf: Since $|A \cup B| > \lambda_j$, for all $w \in C_T$,

$\exists\, a, b \in A \cup B$ s.t. a and b are in the same row of wT. But then

$$S^-_{A\cup B}\{wT\} = * \underbrace{[(1) - (a,b)]\{wT\}}_{0} = 0.$$

Thus, $S^-_{A\cup B}e_T = 0$ also.

Now,

$$S^-_{A\cup B} = \sum_{w\in S_{A\cup B}} (-1)^w w = \sum_{\sigma\in S_{A\cup B}/S_A\times S_B} (-1)^\sigma \sigma \sum_{w\in S_A\times S_B} (-1)^w w$$

$$= g_{A,B}(S_A\times S_B)^-,$$

So $g_{A,B}(S_A\times S_B)^- e_T = 0$, and the result will follow if $(S_A\times S_B)^- e_T$ is a nonzero multiple of e_T.

Since $S_A\times S_B \subseteq C_T$, if $w\in S_A\times S_B$, then

$$(-1)^w w e_T = (-1)^w \kappa_T\{T\} = \kappa_T\{T\}e_T,$$

so $(S_A\times S_B)^- e_T = |S_A\times S_B| e_T$. $\square$

We'll use <u>column tabloids</u>

$$[T] := C_T T$$

e.g. $\left[\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}\right] = \left\{\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}, \begin{smallmatrix} 3 & 2 \\ 1 & \end{smallmatrix}\right\} = \left|\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right|\begin{smallmatrix} 2 \\ \end{smallmatrix}\right|$

Dominance order induced from the map $[T] \mapsto \{T'\}$ and similar facts hold (e.g. dominance lemma)

Pf of Thm 33:

We've already proved linear independence, just need to prove span $E^\lambda = S^\lambda$.

Induction of column tabloid dominance order.

Let $T_0 = \begin{matrix} 1 & a+1 & b+1 & \cdots \\ 2 & \vdots & \vdots & \\ 3 & b & & \\ \vdots & & & \\ a & & & \end{matrix}$ be the tableau numbered

by columns. T_0 is std. and $[T_0] \trianglerighteq [T]$ $\forall T$.

Fix a tableau T. If $S \in T$, then $e_S = \pm e_T$, so we can assume T has increasing columns.

Assume $e_S \in$ span E^λ whenever $[S] \trianglerighteq [T]$.

If T std., done. Otherwise, T has a descent along a row:

$$\begin{array}{ccccc} & a_1 & & b_1 & \\ & \vdots & & \vdots & B \\ & & & \wedge & \\ & a_i & > & b_i & \\ A & \wedge & & \vdots & \\ & \vdots & & \vdots & \\ & & & b_q & \\ & a_p & & & \end{array}$$

By Prop 37, $g_{A,B}\, e_T = 0$, so

$$e_T = -\sum_{\substack{j \\ \sigma_j \neq 1}} (-1)^{\sigma_j} e_{\sigma_j T} \qquad (*)$$

Now, each σ_j is the product of permutations of A, permutations of B, and transpositions (a_i, b_j) w/ $a_i > b_j$.

By the dominance lemma for column tabloids, $[\sigma_j T] \rhd T$ for $\sigma_j \neq 1$, so by induction, $e_T \in \operatorname{Span} E^\lambda$. $\square$

Let f_λ be the number of std. tableaux of shape λ.

Cor 38:

a) $\dim S^\lambda = f^\lambda$

b) $\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$

c) With respect to the basis E^λ, the matrices for the repn S^λ have integer entries.

(Young's natural repn)

d) The character table for S_n has integer entries.

Pf: a) $f^\lambda = |E^\lambda|$.

b) Apply Cor 14: $|G| = \sum_{V:\text{irrep}} (\dim V)^2$

c) $w e_T = e_{wT}$, and by (*) each polytabloid is an integer linear comb. of std. polytabloids.

d) The trace of an integer matrix is an integer. $\square$