Announcements

Lecture 9 video posted: repn theory of $GL_2(\mathbb{F}_q)$

HW2 updated w/ additional problems (due Wed. 2/25)

Lecture 8: partitions and tableaux

Today: Specht modules [Sagan 2.3] [James Ch. 4]

Let T be any tableau of shape λ w/ entries (exactly) $1, 2, \dots, n$ (not necessarily standard)

Recall the row and column stabilizers:

$$R_T := \{ w \in S_n \mid w \text{ preserves the rows of } T \}$$

$$C_T := \{ w \in S_n \mid w \text{ preserves the cols. of } T \}$$

Def 24: Call two tableau T, T' of the same shape λ (row) equivalent, $T \sim T'$, if $T' \in R_T T$

A (λ-)<u>tabloid</u> is an equivalence class

$$\{T\} := R_T T = \{ T' \mid T' \sim T \}$$

"Tableaux w/ unordered row entries"

There is an S_n-action on the set of λ-tabloids given by $w \cdot \{T\} := \{wT\}$.

e.g.

$\overline{\underline{1\,4\,5}}$
$\underline{2\,3}$

We define M^λ to be the S_n-permutation repn assoc. to this action.

Claim: This action is well-defined.

Pf: If $T \sim T'$, we need to show that $\omega T \sim \omega T'$, so that $\{\omega T\} = \{\omega T'\}$. Let $\sigma T = T'$ with $\sigma \in R_T$.

[e.g. $\omega = (13)$

$$T = \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \sim \begin{array}{cc} 2 & 1 \\ 3 & \end{array} = T'$$

$$\omega T = \begin{array}{cc} 3 & 2 \\ 1 & \end{array} \sim \begin{array}{cc} 2 & 3 \\ 1 & \end{array} = \omega T'$$
]

Then $R_{\omega T} = \omega R_T \omega^{-1}$, so $\omega \sigma \omega^{-1} \in R_{\omega T}$, and $\omega \sigma \omega^{-1}(\omega T) = \omega \sigma T = \omega T'$, so $\omega T \sim \omega T'$. $\square$

Ex:

a) $\lambda = \begin{array}{|c|c|} \hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ $\quad M^\lambda = \mathbb{C}\left[\left\{\begin{array}{cc} 1 & 2 \\ 3 & \end{array}\right\}, \left\{\begin{array}{cc} 1 & 3 \\ 2 & \end{array}\right\}, \left\{\begin{array}{cc} 2 & 3 \\ 1 & \end{array}\right\}\right]$

$$\begin{tikzcd}
\left\{\begin{array}{cc} 1 & 2 \\ 3 & \end{array} \sim \begin{array}{cc} 2 & 1 \\ 3 & \end{array}\right\} \arrow[loop left, "(12)"] \arrow[rr, shift left, "(13)\ (123)"] \arrow[dr, shift left, "(23)\ (132)"] & & \left\{\begin{array}{cc} 2 & 3 \\ 1 & \end{array} \sim \begin{array}{cc} 3 & 2 \\ 1 & \end{array}\right\} \arrow[loop right, "(23)"] \arrow[ll, shift left, "(13)\ (132)"] \arrow[dl, shift left, "(12)\ (123)"] \\
& \left\{\begin{array}{cc} 1 & 3 \\ 2 & \end{array} \sim \begin{array}{cc} 3 & 1 \\ 2 & \end{array}\right\} \arrow[loop below, "(13)"] \arrow[ul, shift left, "(23)\ (123)"] \arrow[ur, shift left, "(132)\ (12)"] &
\end{tikzcd}$$

b) $M^{(n)}$ is the trivial repn. $\qquad \begin{array}{c}\hline 1\,2\,\cdots\,n \\ \hline\end{array}$

c) $M^{(1^n)}$ is the regular repn. $\qquad \begin{array}{c}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline\end{array}$

d) $M^{(n-1,n)}$ is the perm. repn. of S_n on $\mathbb{C}^n$

$$\begin{array}{l}\hline 1\ 2\ \cdots (a-1)\,(a+1) \cdots\ n \\ \hline a \\ \hline\end{array}$$

Notice that this is not irred: it has the S_n-invariant subspace spanned by $\displaystyle\sum_a \begin{array}{c}\hline \cdots \\ \hline a \\ \hline\end{array}$

Def 25: A G-repn V is <u>cyclic</u> if $\exists v \in V$ s.t.

$$V = \mathbb{C}[G]v.$$

We say V is <u>generated</u> by v.

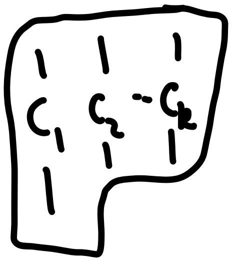
Note: V irred. $\iff$ V is gen'd by v $\forall v \in V$.

Prop 26: M^λ is cyclic, and all tabloids are generators. We have $\dim M^\lambda = \dfrac{n!}{\lambda!}$ where $\lambda! = \lambda_1!\,\lambda_2! \cdots \lambda_{\ell(\lambda)}!$.

Pf: Since S_n acts transitively on $1, \ldots, n$, it acts transitively on all tableaux – hence, tabloids – w/ these entries. We can get from $\{T\}$ to any basis elt. of M^λ, and thus to any linear. comb. of these basis elts. The last sentence is because $|R_T| = \lambda!$. $\square$

Def 27: let $$k_T := \sum_{w \in C_T} (-1)^w \, w \in \mathbb{C}[S_n].$$

$$= k_{c_1} k_{c_2} \cdots k_{c_k} \quad \text{if } T =$$



The <u>polytabloid</u> assoc. to T is

$$e_T := k_T \{T\}.$$

Remark: e_T depends on T, not just $\{T\}$.

e.g. $T = \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$ $\qquad$ $T' = \begin{array}{cc} 2 & 1 \\ 3 & \end{array}$

$$\{T\} = \begin{array}{c} \hline 1\,2 \\ \hline 3 \\ \hline \end{array} = \{T'\}$$

$k_T = () - (13)$ $\qquad$ $k_{T'} = () - (23)$

$$e_T = \begin{array}{c} \hline 1\,2 \\ \hline 3 \\ \hline \end{array} - \begin{array}{c} \hline 2\,3 \\ \hline 1 \\ \hline \end{array} \qquad e_{T'} = \begin{array}{c} \hline 1\,2 \\ \hline 3 \\ \hline \end{array} - \begin{array}{c} \hline 1\,3 \\ \hline 2 \\ \hline \end{array}$$

Def 28: The <u>Specht module</u> S^λ is the submodule of M^λ spanned by polytabloids.

Prop 29: S^λ is a cyclic S_n-module, generated by any polytabloid.

Pf: We prove this by showing that $we_T = e_{wT}$.
Using Def. 27,

$$e_{wT} = K_{wT}\{wT\} = \sum_{u \in C_{wT}} (-1)^u u\{wT\}$$

$$= \sum_{u \in wC_Tw^{-1}} (-1)^u uw\{T\}$$

$$= \sum_{u' \in C_T} (-1)^{u'} wu'\{T\} \qquad (u = wu'w^{-1})$$

$$= wK_T\{T\}$$

$$= we_T$$

$\square$

Ex:

a) $\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

$$e_{\substack{12\\3}} = \begin{array}{l}\hline 1\,2\\ \hline 3\\ \hline\end{array} - \begin{array}{l}\hline 2\,3\\ \hline 1\\ \hline\end{array} = -e_{\substack{32\\1}}$$

$$e_{\substack{13\\2}} = \begin{array}{l}\hline 1\,3\\ \hline 2\\ \hline\end{array} - \begin{array}{l}\hline 2\,3\\ \hline 1\\ \hline\end{array} = -e_{\substack{23\\1}} \qquad e_{\substack{21\\3}} = e_{\substack{12\\3}} + e_{\substack{13\\2}}$$

$$e_{\substack{21\\3}} = \begin{array}{l}\hline 1\,2\\ \hline 3\\ \hline\end{array} - \begin{array}{l}\hline 1\,3\\ \hline 2\\ \hline\end{array} = -e_{\substack{31\\2}}$$

So $S^\lambda = \mathbb{C}\left[e_{\substack{12\\3}}, e_{\substack{13\\2}}\right]$

b) $S^{(n)} = M^{(n)}$ is the trivial repn.

c) $S^{(1^n)}$ is the sign repn. since if $T = \begin{array}{|c|}\hline a_1\\ \hline a_2\\ \hline \vdots\\ \hline a_n\\ \hline\end{array}$,

then $e_T = \sum_{w \in S_n} (-1)^w \left(\begin{array}{c}\hline a_{w(1)}\\ \hline a_{w(2)}\\ \hline \vdots\\ \hline a_{w(n)}\\ \hline\end{array} \right) = \pm e_{\substack{1\\2\\\vdots\\n}}$

d) $S^{(n-1,n)}$ is the submodule of $M^{(n-1,1)}$ spanned by $\{e_{ik}, i<k\}$ where

$$e_{ik} := e_{\substack{i \cdots \\ k}} = \begin{array}{l} \hline 1 \cdots (k-1)\,(k+1) \cdots n \\ \hline k \\ \hline \end{array} - \begin{array}{l} \hline 1 \cdots (i-1)\,(i+1) \cdots n \\ \hline i \\ \hline \end{array}$$

This is the <u>reflection repn</u> of S_n.

We have $S^{(n-1,1)} \oplus \text{triv. repn} = M^{(n-1,1)}$.