Math 506: Group Representation Theory

Lecture: MWF 10:00 - 10:50am
English Bldg. 131

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Course website:

andyhardt.github.io/506_S26/course_page.html

Homework due ~biweekly (see syllabus for more)

Def: A <u>representation</u> is a linear action of a group $\underbrace{}_{}$ (or algebraic object) on a vector space

Equivalently, it is a homom. $\rho : G \to GL(V) = GL_n(F)$

A rep'n V is <u>irreducible</u> if whenever $W \leq V$, $GW \subseteq W$, we have $W = \{0\}$ or $W = V$

Main problem of representation theory: Classify all irreps. of G, and describe how arbitrary repns decompose into irreps.

Ex 1:

$$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$$ group under multiplication

$$L^2(S^1) = \left\{ f : S^1 \to \mathbb{C} \;\middle|\; \int_{S^1} |f(x)|^2 dx < \infty \right\}$$

this is a repn via

$$(z \cdot f)(w) := f(wz)$$

Abelian group $\rightsquigarrow$ irreps are 1D

$$\rho_n : S^1 \longrightarrow \mathbb{C}^* = V_n$$

$n \in \mathbb{Z}$

$$z \longmapsto z^n \in S^1$$

Peter-Weyl Theorem $\rightsquigarrow$ $L^2(G)$ decomposes as an orthogonal direct sum of irreps:

$$L^2(S^1) \cong \widehat{\bigoplus_{n \in \mathbb{Z}}} V_n$$

(Inner prod: $\langle f, g \rangle = \int_{S^1} f(x) \overline{g}(x)\, dx$)

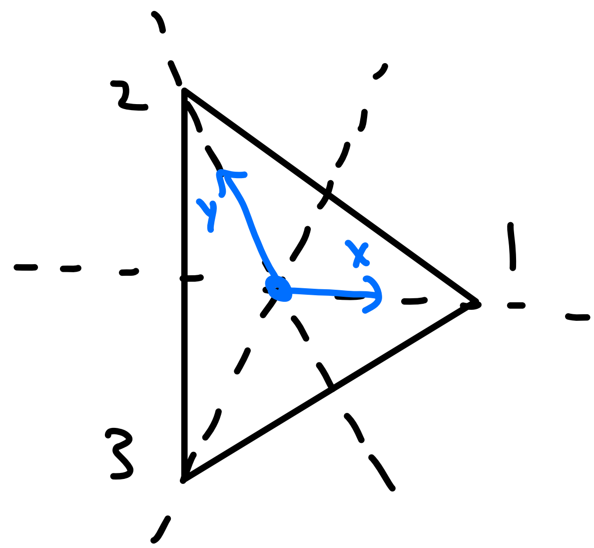
$$f(x) := \underset{\in L^2(S^1)}{g}(e^{2\pi i x}) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$$

Fourier series!

$$\text{where } c_n = \langle f, e^{-2\pi i n\theta}\rangle = \int_0^1 f(x)e^{-2\pi i n x}\,dx$$

Generalizing this picture leads to <u>harmonic analysis</u>

Ex 2: $G = S_3$, $F = \mathbb{C}$



Irreps:

$\rho_{triv}: w \longmapsto [1]$

$\rho_{sgn}: w \longmapsto [(-1)^w]$

$\rho_{ref}: () \longmapsto \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ $\quad (12) \longmapsto \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$

$(13) \longmapsto \begin{bmatrix} -1 & \\ -1 & 1 \end{bmatrix}$ $\quad (23) \longmapsto \begin{bmatrix} 1 & -1 \\ & -1 \end{bmatrix}$

$(123) \longmapsto \begin{bmatrix} & -1 \\ 1 & -1 \end{bmatrix}$ $\quad (132) \longmapsto \begin{bmatrix} -1 & 1 \\ -1 & \end{bmatrix}$

Consider the <u>regular repn</u>

$$V_{reg} = \{ V_{()}, V_{(12)}, V_{(13)}, V_{(23)}, V_{(123)}, V_{(132)} \}$$

w/ the action

$$\omega \cdot V_u := V_{\omega u}$$

We will see later:

$$V_{reg} \cong V_{triv} \oplus V_{sgn} \oplus V_{ref} \oplus V_{ref}$$

decomposes as a direct sum of irreps.

Ex 3: $G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle$, $V = \mathbb{F}_p^2$

$$g^a \mapsto \begin{bmatrix} 1 & a \\ & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ & 1 \end{bmatrix}$$

$W = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$ is invariant since

$$\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} \in W$$

but no other subspace is since

$$\begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ay \\ y \end{bmatrix} \notin \left\langle \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle \text{ if } \begin{matrix} a \neq 0 \\ y \neq 0 \end{matrix}$$

So V <u>cannot</u> be written as the direct sum of irreps!

Ex 4: Def: A <u>quiver repn</u> is a directed graph of vector spaces and linear maps

e.g.

$$\mathbb{C} \xrightarrow{\begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}} \mathbb{C}^3 \xrightarrow{\begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{bmatrix}} \mathbb{C}^2$$

Here, we have two types of "atomic object":

Simples: $\mathbb{C} \longrightarrow 0 \longrightarrow 0$

$0 \longrightarrow \mathbb{C} \longrightarrow 0 \qquad 0 \longrightarrow 0 \longrightarrow \mathbb{C}$

Indecomposibles: also include

$\mathbb{C} \xrightarrow{[1]} \mathbb{C} \longrightarrow 0 \qquad 0 \longrightarrow \mathbb{C} \xrightarrow{[1]} \mathbb{C}$

$$\mathbb{C} \xrightarrow{[1]} \mathbb{C} \xrightarrow{[1]} \mathbb{C}$$

Quiver repns are closely related to the repn theory of f.d. algebras

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Rough course plan:

Sources:

1) Repn theory of finite gps. — Fulton-Harris

2) Repn theory of symmetric gps. — Fulton-Harris, Sagan

3) Repn theory of Lie gps./algebras — Bump, Humphreys, many others

4) Other topics

Many great books/notes on repn. theory. No one source is (or can be) comprehensive. Will post several to the course website throughout the semester.