Math 506, Spring 2026 – Homework 5

Due: Wednesday, May 6th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. All assertions require proof unless otherwise stated. Typesetting your homework using LaTeX is recommended. For this homework, unless otherwise stated all groups are finite, all Lie algebras are complex and finite dimensional, and all representations are finite dimensional and complex.

1. Recall the Frobenius characteristic map $\mathrm{ch} : R = \bigoplus_k R^k \to \Lambda$, where R^k is the space of class functions on S_k, and the product on R is defined by

   $$\chi \cdot \psi = \mathrm{Ind}_{S_k \times S_m}^{S_{k+m}} (\chi \otimes \psi)$$

   for $\chi \in R^k$, $\psi \in R^m$. Recall also that $\mathrm{ch}(f) = \sum_{\mu \vdash n} z_\mu^{-1} f_\mu \, p_\mu$ for $f \in R^n$, where f_μ is the value of f on elements of cycle type μ.

   (a) Prove Theorem 79(d): $\mathrm{ch} : R \to \Lambda$ is a ring homomorphism, where Λ has its usual multiplication of symmetric functions. That is, for $\chi \in R^k$ and $\psi \in R^m$, show that

   $$\mathrm{ch}(\chi \cdot \psi) = \mathrm{ch}(\chi)\mathrm{ch}(\psi).$$

   (*Hint: We can extend the concept of class functions to take values in* Λ *(or any other* $\mathbb{C}$*-algebra). The Hermitian inner-product from Lecture 5 extends coefficient-wise, and restriction, induction, and Frobenius reciprocity extend to these generalized class functions by linearity: for* $H \leq G$*, a class function* χ *on* G*, and any* $\psi : H \to \Lambda$*,*

   $$(\chi, Ind_H^G \psi)_G = (Res_H^G \chi, \psi)_H.$$

   Furthermore, $\mathrm{ch}(f) = (\overline{p}, f) = (p, f)$ *where* $p : S_n \to \Lambda$ *sends* w *to* p_μ *for* w *of cycle type* μ*.*)

   (b) The *Littlewood–Richardson coefficients* $c_{\mu\nu}^\lambda$, for $|\mu| + |\nu| = |\lambda|$, are defined as the tensor product multiplicities of polynomial $\mathrm{GL}_n(\mathbb{C})$-representations:

   $$V^\mu \otimes V^\nu \cong \bigoplus_\lambda (V^\lambda)^{\oplus c_{\mu\nu}^\lambda}.$$

   (Since the character of V^λ is s_λ, these are also the structure constants for the Schur basis: $s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$.) Using part (a), show that as S_n representations

   $$\mathrm{Ind}_{S_k \times S_m}^{S_n} (S^\mu \otimes S^\nu) \cong \bigoplus_\lambda (S^\lambda)^{\oplus c_{\mu\nu}^\lambda},$$

and

$$\mathrm{Res}^{S_n}_{S_k\times S_m} S^\lambda \cong \bigoplus_{\mu,\nu} (S^\mu\otimes S^\nu)^{\oplus c^\lambda_{\mu\nu}}.$$

where $k=|\mu|$, $m=|\nu|$, and $n=k+m$. (Don't get confused between internal and external tensor product; the latter occurs when we have a direct product of groups.)

2. The *Kronecker coefficients* $g(\lambda,\mu,\nu)$, for $\lambda,\mu,\nu\vdash k$, are the tensor product multiplicities of S_k-representations:

$$S^\lambda\otimes S^\mu \cong \bigoplus_{\nu\vdash k} (S^\nu)^{\oplus\, g(\lambda,\mu,\nu)}.$$

The goal of this problem is to give a GL_n-interpretation of these coefficients via Schur–Weyl duality. Throughout, let $V=\mathbb{C}^m$ with $m\geq k$, and $W=V\otimes V=\mathbb{C}^{m^2}$.

(a) Apply Schur–Weyl duality to $V^{\otimes k}\otimes V^{\otimes k}$ to show that, as $(\mathrm{GL}(V)\times\mathrm{GL}(V)\times S_k)$-representations,

$$V^{\otimes k}\otimes V^{\otimes k} \cong \bigoplus_{\lambda,\mu,\nu\vdash k} g(\lambda,\mu,\nu)\; V^\lambda\otimes V^\mu\otimes S^\nu,$$

where on the left side, the S_k acts diagonally on the two $V^{\otimes k}$ factors, and on the right side, the two $\mathrm{GL}(V)$-factors act on V^λ and V^μ respectively.

(b) Let $\mathrm{GL}(V)\times\mathrm{GL}(V)$ sit inside $\mathrm{GL}(W)$ via the Kronecker product $(A,B)\mapsto A\otimes B$. Show that the map $V^{\otimes k}\otimes V^{\otimes k}\xrightarrow{\sim} W^{\otimes k}$

$$(v_1\otimes\cdots\otimes v_k)\otimes(w_1\otimes\cdots\otimes w_k)\mapsto(v_1\otimes w_1)\otimes\cdots\otimes(v_k\otimes w_k)$$

is an $(S_k\times\mathrm{GL}(V)\times\mathrm{GL}(V))$-equivariant isomorphism.

(c) Using Schur-Weyl duality for $W^{\otimes k}$ and parts (a) and (b), conclude that as $\mathrm{GL}(V)\times\mathrm{GL}(V)$-representations,

$$\mathrm{Res}^{\mathrm{GL}(W)}_{\mathrm{GL}(V)\times\mathrm{GL}(V)} W^\nu \cong \bigoplus_{\lambda,\mu\vdash k} g(\lambda,\mu,\nu)\; V^\lambda\otimes V^\mu.$$

In particular,

$$g(\lambda,\mu,\nu)=\dim\mathrm{Hom}_{\mathrm{GL}(V)\times\mathrm{GL}(V)}(V^\lambda\otimes V^\mu,\, W^\nu).$$

3. Let $\mathfrak{g}$ be a complex Lie algebra. The *Killing form* is the symmetric bilinear form $B:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ defined by $B(X,Y)=\mathrm{tr}(\mathrm{ad}_X\circ\mathrm{ad}_Y)$.

(a) Show that B is Ad-invariant: $B(\mathrm{Ad}_A(X),\mathrm{Ad}_A(Y))=B(X,Y)$ for all $A\in G$, $X,Y\in\mathfrak{g}$. (*Hint: first show that* $ad_{Ad_A(X)}=Ad_A\circ ad_X\circ Ad_A^{-1}$ *as operators on* $\mathfrak{g}$.)

(b) Compute B explicitly for $\mathfrak{sl}_2(\mathbb{C})$ with standard basis e, f, h satisfying $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

4. Let $Z = (z_{ij})_{1 \le i,j \le n}$ be an $n \times n$ matrix of indeterminates, and let $\mathrm{GL}_n(\mathbb{C})$ act on $\mathbb{C}[z_{ij}]$ by $(g \cdot f)(Z) = f(Zg)$. For $S \subseteq [n]$ with $|S| = k$, write

$$p_S = \det(z_{ij})_{1 \le i \le k,\, j \in S}$$

for the minor of Z using rows $1, \dots, k$ and columns S. Let $N \le \mathrm{GL}_n(\mathbb{C})$ denote the subgroup of upper triangular matrices with 1's on the diagonal.

(a) Show that $p_{[k]}$ is a highest weight vector of weight $\varepsilon_1 + \cdots + \varepsilon_k$ (recall ε_i is the ith fundamental weight), and that the subrepresentation generated by $p_{[k]}$ is $\mathrm{span}\{p_S : |S| = k\} \cong \bigwedge^k(\mathbb{C}^n)$.

(*Hint: Use the Cauchy–Binet formula.*)

(b) For a partition $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$ of length $\le n$, define

$$\Delta_\lambda = \prod_{k=1}^{n} p_{[k]}^{\lambda_k - \lambda_{k+1}}.$$

Show that Δ_λ is a highest weight vector of weight λ, and that the GL_n-subrepresentation of $\mathbb{C}[z_{ij}]$ it generates is isomorphic to V^λ.

(*Hint: use Cauchy–Binet to show the polynomial* GL_n*-representation generated by* Δ_λ *is finite-dimensional, hence completely reducible, and apply Part 6 of the proof sketch of Theorem 57, which you can assume holds equally well for* GL_n*.*)

(c) Take $n = 3$ and $\lambda = (2, 1, 0)$. By part (b), $V^{(2,1,0)}$ is the GL_3-representation generated by $\Delta_{(2,1,0)} = p_{\{1\}} \cdot p_{\{1,2\}} = z_{11}(z_{11}z_{22} - z_{12}z_{21})$. Write out a basis for $V^{(2,1,0)}$ explicitly as polynomials in the z_{ij}.

(*Hint: by the Cauchy–Binet expansion from part (b),* $V^{(2,1,0)}$ *is contained in the span of the* $3 \times 3 = 9$ *products* $p_{\{s\}} \cdot p_{\{t,u\}}$ *for* $s \in [3]$, $\{t, u\} \subseteq [3]$. *Find the linear relations among these nine products (you don't need a rigorous argument that you've found them all), and verify that the dimension equals that of* $\dim V^{(2,1,0)}$*.*)