

Math 506, Spring 2026 – Homework 5

Due: Wednesday, May 6th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. All assertions require proof unless otherwise stated. Typesetting your homework using LaTeX is recommended. For this homework, unless otherwise stated all groups are finite, all Lie algebras are complex and finite dimensional, and all representations are finite dimensional and complex.

- Recall the Frobenius characteristic map $\text{ch} : R = \bigoplus_k R^k \rightarrow \Lambda$, where R^k is the space of class functions on S_k , and the product on R is defined by

$$\chi \cdot \psi = \text{Ind}_{S_k \times S_m}^{S_{k+m}} (\chi \otimes \psi)$$

for $\chi \in R^k$, $\psi \in R^m$. Recall also that $\text{ch}(f) = \sum_{\mu \vdash n} z_\mu^{-1} f_\mu p_\mu$ for $f \in R^n$, where f_μ is the value of f on elements of cycle type μ .

- Prove Theorem 79(d): $\text{ch} : R \rightarrow \Lambda$ is a ring homomorphism, where Λ has its usual multiplication of symmetric functions. That is, for $\chi \in R^k$ and $\psi \in R^m$, show that

$$\text{ch}(\chi \cdot \psi) = \text{ch}(\chi)\text{ch}(\psi).$$

(Hint: We can extend the concept of class functions to take values in Λ (or any other \mathbb{C} -algebra). The Hermitian inner-product from Lecture 5 extends coefficient-wise, and restriction, induction, and Frobenius reciprocity extend to these generalized class functions by linearity: for $H \leq G$, a class function χ on G , and any $\psi : H \rightarrow \Lambda$,

$$(\chi, \text{Ind}_H^G \psi)_G = (\text{Res}_H^G \chi, \psi)_H.$$

Furthermore, $\text{ch}(f) = (\bar{p}, f) = (p, f)$ where $p : S_n \rightarrow \Lambda$ sends w to p_μ for w of cycle type μ .)

- The Littlewood–Richardson coefficients $c_{\mu\nu}^\lambda$, for $|\mu| + |\nu| = |\lambda|$, are defined as the tensor product multiplicities of polynomial $\text{GL}_n(\mathbb{C})$ -representations:

$$V^\mu \otimes V^\nu \cong \bigoplus_{\lambda} (V^\lambda)^{\oplus c_{\mu\nu}^\lambda}.$$

(Since the character of V^λ is s_λ , these are also the structure constants for the Schur basis: $s_\mu s_\nu = \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda$.) Using part (a), show that as S_n representations

$$\text{Ind}_{S_k \times S_m}^{S_n} (S^\mu \otimes S^\nu) \cong \bigoplus_{\lambda} (S^\lambda)^{\oplus c_{\mu\nu}^\lambda},$$

and

$$\text{Res}_{S_k \times S_m}^{S_n} S^\lambda \cong \bigoplus_{\mu, \nu} (S^\mu \otimes S^\nu)^{\oplus c_{\mu\nu}^\lambda}.$$

where $k = |\mu|$, $m = |\nu|$, and $n = k + m$. (Don't get confused between internal and external tensor product; the latter occurs when we have a direct product of groups.)

Solution.

- (a) Let $n = k + m$. By the definition of ch and the generalized Frobenius reciprocity from the hint,

$$\text{ch}(\chi \cdot \psi) = (p, \text{Ind}_{S_k \times S_m}^{S_n} (\chi \otimes \psi)) = (\text{Res}_{S_k \times S_m}^{S_n} p, \chi \otimes \psi).$$

For any permutation w , let $\text{cyc}(w)$ denote the partition which is the cycle type of w . For $(\sigma, \tau) \in S_k \times S_m$, the cycle type of (σ, τ) in S_n is the union of the cycle types of σ and τ , so $(\text{Res}_{S_k \times S_m}^{S_n} p)(\sigma, \tau) = p_{\text{cyc}(\sigma)} p_{\text{cyc}(\tau)}$. Therefore,

$$\begin{aligned} (\text{Res}_{S_k \times S_m}^{S_n} p, \chi \otimes \psi) &= \frac{1}{k! m!} \sum_{\substack{\sigma \in S_k \\ \tau \in S_m}} p_{\text{cyc}(\sigma)} p_{\text{cyc}(\tau)} \chi(\sigma) \psi(\tau) \\ &= \left(\frac{1}{k!} \sum_{\sigma \in S_k} p_{\text{cyc}(\sigma)} \chi(\sigma) \right) \left(\frac{1}{m!} \sum_{\tau \in S_m} p_{\text{cyc}(\tau)} \psi(\tau) \right) \\ &= \text{ch}(\chi) \text{ch}(\psi). \end{aligned}$$

- (b) Since the character of V^λ is s_λ , characters are multiplicative under tensor product, so $c_{\mu\nu}^\lambda$ as defined via GL_n -representations equals the structure constant in $s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$. Applying ch^{-1} (which is an isomorphism by part (a) and Theorem 79(b)) to both sides, and using $\text{ch}(\chi^\lambda) = s_\lambda$,

$$\chi^\mu \cdot \chi^\nu = \sum_\lambda c_{\mu\nu}^\lambda \chi^\lambda.$$

Since characters determine representations, this gives

$$\text{Ind}_{S_k \times S_m}^{S_n} (S^\mu \otimes S^\nu) \cong \bigoplus_\lambda (S^\lambda)^{\oplus c_{\mu\nu}^\lambda}.$$

By (the original version of) Frobenius reciprocity, the multiplicity of $S^\mu \otimes S^\nu$ in $\text{Res}_{S_k \times S_m}^{S_n} S^\lambda$ equals the multiplicity of S^λ in $\text{Ind}_{S_k \times S_m}^{S_n} (S^\mu \otimes S^\nu)$, which is $c_{\mu\nu}^\lambda$. Therefore $\text{Res}_{S_k \times S_m}^{S_n} S^\lambda \cong \bigoplus_{\mu, \nu} (S^\mu \otimes S^\nu)^{\oplus c_{\mu\nu}^\lambda}$.

2. The *Kronecker coefficients* $g(\lambda, \mu, \nu)$, for $\lambda, \mu, \nu \vdash k$, are the tensor product multiplicities of S_k -representations:

$$S^\lambda \otimes S^\mu \cong \bigoplus_{\nu \vdash k} (S^\nu)^{\oplus g(\lambda, \mu, \nu)}.$$

The goal of this problem is to give a GL_n -interpretation of these coefficients via Schur–Weyl duality. Throughout, let $V = \mathbb{C}^m$ with $m \geq k$, and $W = V \otimes V = \mathbb{C}^{m^2}$.

- (a) Apply Schur–Weyl duality to $V^{\otimes k} \otimes V^{\otimes k}$ to show that, as $(\mathrm{GL}(V) \times \mathrm{GL}(V) \times S_k)$ -representations,

$$V^{\otimes k} \otimes V^{\otimes k} \cong \bigoplus_{\lambda, \mu, \nu \vdash k} g(\lambda, \mu, \nu) V^\lambda \otimes V^\mu \otimes S^\nu,$$

where on the left side, the S_k acts diagonally on the two $V^{\otimes k}$ factors, and on the right side, the two $\mathrm{GL}(V)$ -factors act on V^λ and V^μ respectively.

- (b) Let $\mathrm{GL}(V) \times \mathrm{GL}(V)$ sit inside $\mathrm{GL}(W)$ via the Kronecker product $(A, B) \mapsto A \otimes B$. Show that the map $V^{\otimes k} \otimes V^{\otimes k} \xrightarrow{\sim} W^{\otimes k}$

$$(v_1 \otimes \cdots \otimes v_k) \otimes (w_1 \otimes \cdots \otimes w_k) \mapsto (v_1 \otimes w_1) \otimes \cdots \otimes (v_k \otimes w_k)$$

is an $(S_k \times \mathrm{GL}(V) \times \mathrm{GL}(V))$ -equivariant isomorphism.

- (c) Using Schur–Weyl duality for $W^{\otimes k}$ and parts (a) and (b), conclude that as $\mathrm{GL}(V) \times \mathrm{GL}(V)$ -representations,

$$\mathrm{Res}_{\mathrm{GL}(V) \times \mathrm{GL}(V)}^{\mathrm{GL}(W)} W^{\otimes k} \cong \bigoplus_{\lambda, \mu \vdash k} g(\lambda, \mu, \nu) V^\lambda \otimes V^\mu.$$

In particular,

$$g(\lambda, \mu, \nu) = \dim \mathrm{Hom}_{\mathrm{GL}(V) \times \mathrm{GL}(V)}(V^\lambda \otimes V^\mu, W^\nu).$$

Solution.

- (a) By Schur–Weyl duality, $V^{\otimes k} \cong \bigoplus_{\lambda \vdash k} V^\lambda \otimes S^\lambda$ as $(\mathrm{GL}(V) \times S_k)$ -representations, so

$$V^{\otimes k} \otimes V^{\otimes k} \cong \bigoplus_{\lambda, \mu} V^\lambda \otimes V^\mu \otimes (S^\lambda \otimes S^\mu) \cong \bigoplus_{\lambda, \mu, \nu} g(\lambda, \mu, \nu) V^\lambda \otimes V^\mu \otimes S^\nu,$$

using $S^\lambda \otimes S^\mu \cong \bigoplus_{\nu} g(\lambda, \mu, \nu) S^\nu$ in the last step.

- (b) The map sends a pure tensor to a pure tensor, so it is well-defined and clearly bijective. For S_k -equivariance: $\sigma \in S_k$ acts on the left side by simultaneously permuting the factors of both $V^{\otimes k}$'s, sending $(v_1 \otimes \cdots \otimes v_k) \otimes (w_1 \otimes \cdots \otimes w_k)$ to $(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}) \otimes (w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(k)})$, which maps to $(v_{\sigma^{-1}(1)} \otimes w_{\sigma^{-1}(1)}) \otimes \cdots \otimes (v_{\sigma^{-1}(k)} \otimes w_{\sigma^{-1}(k)})$; this is exactly σ acting on $(v_1 \otimes w_1) \otimes \cdots \otimes (v_k \otimes w_k)$ in $W^{\otimes k}$. For $\mathrm{GL}(V) \times \mathrm{GL}(V)$: (A, B) sends $v_i \otimes w_i$ to $Av_i \otimes Bw_i = (A \otimes B)(v_i \otimes w_i)$, matching the action of $A \otimes B \in \mathrm{GL}(W)$.

(c) Since $m^2 \geq m \geq k$, Schur–Weyl duality applied to $W \cong \mathbb{C}^{m^2}$ gives

$$W^{\otimes k} \cong \bigoplus_{\nu \vdash k} W^\nu \otimes S^\nu,$$

where W^ν is the irreducible $\mathrm{GL}(W)$ -representation of highest weight ν . Combining with part (b), we identify the decompositions from (a) and the one above as $(\mathrm{GL}(V) \times \mathrm{GL}(V) \times S_k)$ -representations:

$$\bigoplus_{\lambda, \mu, \nu \vdash k} g(\lambda, \mu, \nu) V^\lambda \otimes V^\mu \otimes S^\nu \cong \bigoplus_{\nu \vdash k} W^\nu \otimes S^\nu.$$

Since distinct S^ν are non-isomorphic, the S^ν -isotypic components must match, giving

$$\mathrm{Res}_{\mathrm{GL}(V) \times \mathrm{GL}(V)}^{\mathrm{GL}(W)} W^\nu \cong \bigoplus_{\lambda, \mu \vdash k} g(\lambda, \mu, \nu) V^\lambda \otimes V^\mu$$

as $\mathrm{GL}(V) \times \mathrm{GL}(V)$ -representations. The $V^\lambda \otimes V^\mu$ are irreducible as $\mathrm{GL}(V) \times \mathrm{GL}(V)$ -representations by the same argument as HW2#3, and pairwise non-isomorphic since distinct V^λ have distinct characters. Applying $\mathrm{Hom}_{\mathrm{GL}(V) \times \mathrm{GL}(V)}(V^\lambda \otimes V^\mu, -)$ to both sides gives $g(\lambda, \mu, \nu) = \dim \mathrm{Hom}_{\mathrm{GL}(V) \times \mathrm{GL}(V)}(V^\lambda \otimes V^\mu, W^\nu)$.

3. Let \mathfrak{g} be a complex Lie algebra. The *Killing form* is the symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ defined by $B(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$.

- (a) Show that B is Ad-invariant: $B(\mathrm{Ad}_A(X), \mathrm{Ad}_A(Y)) = B(X, Y)$ for all $A \in G$, $X, Y \in \mathfrak{g}$. (*Hint: first show that $\mathrm{ad}_{\mathrm{Ad}_A(X)} = \mathrm{Ad}_A \circ \mathrm{ad}_X \circ \mathrm{Ad}_A^{-1}$ as operators on \mathfrak{g} .*)
- (b) Compute B explicitly for $\mathfrak{sl}_2(\mathbb{C})$ with standard basis e, f, h satisfying $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

Solution.

- (a) We first claim that $\mathrm{ad}_{\mathrm{Ad}_A(X)} = \mathrm{Ad}_A \circ \mathrm{ad}_X \circ \mathrm{Ad}_A^{-1}$ as operators on \mathfrak{g} . Indeed, for any $Z \in \mathfrak{g}$,

$$\begin{aligned} (\mathrm{Ad}_A \circ \mathrm{ad}_X \circ \mathrm{Ad}_A^{-1})(Z) &= A[X, A^{-1}ZA]A^{-1} \\ &= AXA^{-1}Z - ZAXA^{-1} \\ &= [AXA^{-1}, Z] \\ &= \mathrm{ad}_{\mathrm{Ad}_A(X)}(Z). \end{aligned}$$

Using this, together with the fact that trace is conjugation-invariant,

$$\begin{aligned} B(\mathrm{Ad}_A(X), \mathrm{Ad}_A(Y)) &= \mathrm{tr}(\mathrm{ad}_{\mathrm{Ad}_A(X)} \circ \mathrm{ad}_{\mathrm{Ad}_A(Y)}) \\ &= \mathrm{tr}(\mathrm{Ad}_A \circ \mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathrm{Ad}_A^{-1}) \\ &= \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \\ &= B(X, Y). \end{aligned}$$

(b) We compute the matrices of $\text{ad}_e, \text{ad}_f, \text{ad}_h$ in the ordered basis $\{e, f, h\}$:

$$\text{ad}_e = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{ad}_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{ad}_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since B is symmetric, we need to compute $\text{tr}(\text{ad}_X \circ \text{ad}_Y)$ for each unordered pair $\{X, Y\} \subseteq \{e, f, h\}$. The off-diagonal pairs $\{e, h\}$ and $\{f, h\}$ give zero since $\text{tr}(\text{ad}_e \text{ad}_h) = \text{tr}(\text{ad}_f \text{ad}_h) = 0$ by direct computation. For the remaining entries,

$$\text{ad}_e^2 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{ad}_f^2 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{ad}_h^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{ad}_e \text{ad}_f = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

so $B(e, e) = B(f, f) = 0$, $B(h, h) = 8$, and $B(e, f) = 4$. Therefore, the matrix of B in the basis $\{e, f, h\}$ is

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

In particular, B is nondegenerate (nondegeneracy of Killing form is equivalent to a Lie algebra being semisimple).

4. Let $Z = (z_{ij})_{1 \leq i, j \leq n}$ be an $n \times n$ matrix of indeterminates, and let $\text{GL}_n(\mathbb{C})$ act on $\mathbb{C}[z_{ij}]$ by $(g \cdot f)(Z) = f(Zg)$. For $S \subseteq [n]$ with $|S| = k$, write

$$p_S = \det(z_{ij})_{1 \leq i \leq k, j \in S}$$

for the minor of Z using rows $1, \dots, k$ and columns S . Let $N \leq \text{GL}_n(\mathbb{C})$ denote the subgroup of upper triangular matrices with 1's on the diagonal.

(a) Show that $p_{[k]}$ is a highest weight vector of weight $\varepsilon_k = e_1 + \dots + e_k$ (recall ε_i is the i th fundamental weight), and that the subrepresentation generated by $p_{[k]}$ is $\text{span}\{p_S : |S| = k\} \cong \wedge^k(\mathbb{C}^n)$.

(Hint: Use the Cauchy–Binet formula for minors of a product of matrices.)

(b) For a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0)$ of length $\leq n$, define

$$\Delta_\lambda = \prod_{k=1}^n p_{[k]}^{\lambda_k - \lambda_{k+1}}.$$

Show that Δ_λ is a highest weight vector of weight λ , and that the GL_n -subrepresentation of $\mathbb{C}[z_{ij}]$ it generates is isomorphic to V^λ .

(Hint: use Cauchy–Binet to show the polynomial GL_n -representation generated by Δ_λ is finite-dimensional, hence completely reducible, and apply Part 6 of the proof sketch of Theorem 57, which you can assume holds equally well for GL_n .)

- (c) Take $n = 3$ and $\lambda = (2, 1, 0)$. By part (b), $V^{(2,1,0)}$ is the GL_3 -representation generated by $\Delta_{(2,1,0)} = p_{\{1\}} \cdot p_{\{1,2\}} = z_{11}(z_{11}z_{22} - z_{12}z_{21})$. Write out a basis for $V^{(2,1,0)}$ explicitly as polynomials in the z_{ij} .

(Hint: by the Cauchy–Binet expansion from part (b), $V^{(2,1,0)}$ is contained in the span of the $3 \times 3 = 9$ products $p_{\{s\}} \cdot p_{\{t,u\}}$ for $s \in [3]$, $\{t, u\} \subseteq [3]$. Find the linear relations among these nine products (you don't need a rigorous argument that you've found them all), and verify that the dimension equals that of $\dim V^{(2,1,0)}$.)

Solution.

- (a) For $g = \mathrm{diag}(t_1, \dots, t_n)$, right-multiplying Z by g scales column j by t_j , so $p_{[k]}$ has weight $e_1 + \dots + e_k = \varepsilon_k$. For $u \in N$, right-multiplying by u adds to each column a combination of earlier columns, which does not change $p_{[k]}$ since column operations don't change determinants. Thus, $p_{[k]}$ is a highest weight vector.

By Cauchy–Binet, for any $g \in \mathrm{GL}_n$ and any $S \subseteq [n]$ with $|S| = k$, $p_S(Zg) = \sum_{|T|=k} \det(g_{T \times S}) p_T(Z)$. Since g is a fixed matrix of scalars, $\det(g_{T \times S}) \in \mathbb{C}$, so this is a \mathbb{C} -linear combination of the p_T . Thus $W := \mathrm{span}\{p_S : |S| = k\}$ is GL_n -invariant, and the GL_n -representation generated by $p_{[k]}$ is contained in W . The p_S are linearly independent: for each $S = \{s_1 < \dots < s_k\}$, the monomial $z_{1,s_1} z_{2,s_2} \dots z_{k,s_k}$ appears in p_S but in no other p_T with $T \neq S$. So $\dim W = \binom{n}{k}$. By complete reducibility, W contains the irreducible of highest weight $\varepsilon_1 + \dots + \varepsilon_k$, which is $\bigwedge^k(\mathbb{C}^n)$ of dimension $\binom{n}{k}$. So $W = \bigwedge^k(\mathbb{C}^n)$. The explicit isomorphism $W \rightarrow \bigwedge^k(\mathbb{C}^n)$ is

$$p_S = p_{\{s_1, \dots, s_k\}} \mapsto v_{s_1 \wedge \dots \wedge s_k}.$$

- (b) Since each $p_{[k]}$ is fixed by N , so is Δ_λ . The weight of $p_{[k]}^{\lambda_k - \lambda_{k+1}}$ is $(\lambda_k - \lambda_{k+1})(e_1 + \dots + e_k)$, so Δ_λ has weight $\sum_{k=1}^n (\lambda_k - \lambda_{k+1})(e_1 + \dots + e_k)$. The coefficient of e_i is $\sum_{k \geq i} (\lambda_k - \lambda_{k+1}) = \lambda_i$, so Δ_λ has weight λ .

By Cauchy–Binet,

$$(g \cdot \Delta_\lambda)(Z) = \prod_{k=1}^n \left(\sum_{|S|=k} \det(g_{S \times [k]}) p_S(Z) \right)^{\lambda_k - \lambda_{k+1}}.$$

Expanding, every element of the GL_n -span U of Δ_λ is a \mathbb{C} -linear combination of products $\prod_k \prod_j p_{S_{k,j}}$ using $\lambda_k - \lambda_{k+1}$ size- k minors for each k . These are polynomials of fixed degree $|\lambda|$, so U is finite-dimensional, hence completely reducible. Since Δ_λ generates U and is a highest weight vector of weight λ , U is a highest-weight cyclic representation, hence irreducible by Part 6 of the proof sketch of Theorem 57. So $U \cong V^\lambda$.

(c) We have $\Delta_{(2,1,0)} = p_{\{1\}} \cdot p_{\{1,2\}}$. Using the Cauchy–Binet expansion from the previous problem, $V^{(2,1,0)}$ is contained in

$$W = \text{span}\{p_{\{s\}} \cdot p_{\{t,u\}} : s \in [3], \{t, u\} \subseteq [3]\}.$$

The nine spanning elements satisfy the relation

$$p_{\{1\}} p_{\{2,3\}} - p_{\{2\}} p_{\{1,3\}} + p_{\{3\}} p_{\{1,2\}} = 0, \tag{1}$$

which is verified by expanding: $p_{\{s\}} = z_{1s}$ and $p_{\{t,u\}} = z_{1t}z_{2u} - z_{1u}z_{2t}$, so

$$z_{11}(z_{12}z_{23} - z_{13}z_{22}) - z_{12}(z_{11}z_{23} - z_{13}z_{21}) + z_{13}(z_{11}z_{22} - z_{12}z_{21}) = 0.$$

To see that (1) is the only relation: each product $p_{\{s\}} \cdot p_{\{t,u\}}$ is a degree-3 polynomial in the six variables $z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{23}$, and one verifies by inspection that any eight of the nine products are linearly independent. So $\dim W = 8$.

By the Weyl dimension formula for GL_3 ,

$$\dim V^{(\lambda_1, \lambda_2, \lambda_3)} = \prod_{1 \leq i < j \leq 3} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

which for $\lambda = (2, 1, 0)$ gives $\frac{2}{1} \cdot \frac{4}{2} \cdot \frac{2}{1} = 8$ (alternatively, one can count directly that there are eight semistandard Young tableaux of shape $(2, 1)$ with entries in $\{1, 2, 3\}$). Since $V^{(2,1,0)} \subseteq W$ and both have dimension 8, we conclude $V^{(2,1,0)} = W$. Any 8 of the 9 products $p_{\{s\}} \cdot p_{\{t,u\}}$ form a basis.