Math 506, Spring 2026 – Homework 4

Due: Monday, April 20th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. All assertions require proof unless otherwise stated. Typesetting your homework using LaTeX is recommended. For this homework, unless otherwise stated all groups are finite, all Lie algebras are complex and finite dimensional, and all representations are finite dimensional and complex.

1. Let V be a finite dimensional vector space over a field K. Recall that an operator $A \in \mathrm{End}(V)$ is *diagonalizable* if V has a basis consisting of eigenvectors for A, and a set of operators $\mathcal{A}$ is *simultaneously diagonalizable* if V has a basis such that each basis vector is an eigenvector for all $A \in \mathcal{A}$.

   Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be a set of diagonalizable operators on V. The goal of this problem is to prove that $\mathcal{A}$ is simultaneously diagonalizable if and only if its elements pairwise commute.

   (a) Prove that if $\mathcal{A}$ is simultaneously diagonalizable that its elements must pairwise commute.

   (b) Let $A, B \in \mathrm{End}(V)$ be commuting operators. For an eigenvalue λ of A, let $E_\lambda = \{v \in V | Av = \lambda v\}$ be its λ-eigenspace. Prove that E_λ is a B-invariant subspace.

   (c) Let $A \in \mathrm{End}(V)$, and let $W \leq V$ be an A-invariant subspace. Prove that if A is diagonalizable on V that it is also diagonalizable on W.

   (d) Use the proceding two parts to show that if $\mathcal{A} \subseteq \mathrm{End}(V)$ is a finite collection of diagonalizable operators on V and its elements pairwise commute that $\mathcal{A}$ is simultaneously diagonalizable.

   (e) Finally, show that the previous part holds even when $\mathcal{A}$ is infinite (but V is still finite dimensional).

   Solution.

   (a) If $\mathcal{A}$ is simultaneously diagonalizable, let $v_1, \ldots, v_n$ be a basis of eigenvectors for all $A \in \mathcal{A}$, and let $\lambda_{i,A} \in K$ denote the eigenvalue: $Av_i =: \lambda_{i,A} v_i$. Then $ABv_i = \lambda_{i,A}\lambda_{i,B} v_i = \lambda_{i,B}\lambda_{i,A} v_i = BAv_i$, and since the actions of A and B commute on a basis of V, they commute on all of V.

(b) Let $B \in \mathcal{A}$ and $v \in E_\lambda$. Then since A and B commute,

$$ABv = BAv = B\lambda v = \lambda Bv,$$

so $Bv \in E_\lambda$.

(c) *Method 1:* Using Jordan canonical form, a matrix is diagonalizable if and only if its minimal polynomial splits over K and has no repeated factors since the multiplicity k of a factor $(x - \lambda)$ equals the size of the largest Jordan block with eigenvalue λ. Now, let $m_V(x)$ be the minimal polynomial for A on V, and let $m_W(x)$ be the minimal polynomial for A on W. Since $m_V(A) = 0$, it restricts to the zero operator on W, and so $m_V(A|_W) = 0$. Now, all polynomials p where $p(A|_W) = 0$ are multiples of m_W, so $m_W|m_V$. Thus, if A is diagonalizable, m_V and thus m_W split over K and have no repeated factors, so $A|_W$ is diagonalizable.

Method 2: Assume that A is diagonalizable, with eigenvalues $\lambda_1, \ldots, \lambda_r$. Then V is the direct sum of the A-eigenspaces: $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r}$. If $w \in W$ we can therefore write $w = v_1 + \cdots + v_r$ where $v_i \in E_{\lambda_i}$. We claim that each v_i is in W, which implies that $W = \bigoplus_i (E_{\lambda_i} \cap W)$, and thus $A|_W$ is diagonalizable.

To prove the claim, we use induction on r; if $r = 1$, $v_1 = w \in W$. If $r > 1$, then by W invariance $Aw - \lambda_1 w \in W$ and

$$Aw - \lambda_1 w = (\lambda_2 - \lambda_1)v_2 + \cdots (\lambda_r - \lambda_1)v_r.$$

By the inductive hypothesis, each $(\lambda_i - \lambda_1)v_i \in W$, so since all the eigenvalues are distinct all the $v_i \in W (i \geq 2)$. Then also $v_1 = w - v_2 - \cdots - v_r \in W$, so the claim holds.

(d) Let $\mathcal{A} = \{A_1, \ldots, A_k\}$. We use induction on k. Fix an eigenspace W for A_k. Then A_k acts by a scalar on W, and by part (b), W is also an A_i-invariant subspace for $i = 1, \ldots, k-1$. By part (c), $A_1, \ldots, A_{k-1}$ are diagonalizable on W, so by the inductive hypothesis they are simultaneously diagonalizable on W. As $A_k|_W$ is a scalar matrix, it is diagonal with respect to any basis, so $A_1|_W, \ldots, A_k|_W$ are simultaneously diagonalizable. Since A_k is diagonalizable, V is the direct sum of the A_k-eigenspaces, so by choosing a basis of simultaneous eigenvectors on each A_k-eigenspace, we obtain a basis of V consisting of simultaneous eigenvectors for $\mathcal{A}$, so $\mathcal{A}$ is simultaneously diagonalizable.

(e) Consider the space $U := \text{span}(\mathcal{A}) \subseteq \text{End}(V)$. We claim that U is simultaneously diagonalizable, so $\mathcal{A} \subseteq U$ also is. First, elements of U are finite linear combinations of elements of $\mathcal{A}$. The previous part says that any finite subset of $\mathcal{A}$ is simultaneously diagonalizable, so if $u \in U$ can be expressed as $u = A_1 + \cdots + A_r$, with $A_i \in \mathcal{A}$, there is some basis with respect to which each A_i is diagonal, and therefore u is diagonal with respect to the same basis. We conclude that every element of U is diagonalizable.

Now, since V is finite dimensional, so is $\text{End}(V)$, so choose a basis $A_1, \ldots, A_k$ for U. By the previous part, $A_1, \ldots, A_k$ are simultaneously diagonalizable, and by the reasoning in the previous paragraph, this basis of simultanteous eigenvectors is also a basis of eigenvectors for every linear combination of the $A_1, \ldots, A_k$ i.e. for every $u \in U$. Therefore, U is simultaneously diagonalizable.

2. For each of the irreducible rank 2 root systems, $A_2, B_2(=C_2)$, and G_2, verify explicitly that the Weyl group acts transitively on the sets of roots of a given length.

   Solution. See the diagrams in the Lecture 19 notes. Choosing a representative long/short root for each root system, we exhibit the Weyl group elements sending this root to each root of the same length.

   For A_2, all roots are the same length. Let γ be any root. A solution to $w\alpha = \gamma$ is given by

γ	α	$\alpha+\beta$	β	$-\alpha$	$-\alpha-\beta$	$-\beta$
w	1	s_β	$s_\alpha s_\beta$	s_α	$s_\beta s_\alpha$	$s_\alpha s_\beta s_\alpha$

.

   For $B_2(=C_2)$, first let γ be a long root. A solution to $w\alpha = \gamma$ is given by

γ	α	$\alpha+2\beta$	$-\alpha$	$-\alpha-2\beta$
w	1	s_β	s_α	$s_\beta s_\alpha$

,

   and if γ is a short root, a solution to $w\beta = \gamma$ is

γ	β	$\alpha+\beta$	$-\beta$	$-\alpha-\beta$
w	1	s_α	s_β	$s_\alpha s_\beta$

.

   Finally, for G_2, first let γ be a long root. A solution to $w\alpha = \gamma$ is given by

γ	α	$2\alpha+3\beta$	$\alpha+3\beta$	$-\alpha$	$-2\alpha-3\beta$	$-\alpha-3\beta$
w	1	$s_\alpha s_\beta$	s_β	s_α	$s_\alpha s_\beta s_\alpha$	$s_\beta s_\alpha$

,

   and if γ is a short root, a solution to $w\beta = \gamma$ is

γ	β	$\alpha+2\beta$	$\alpha+\beta$	$-\beta$	$-\alpha-2\beta$	$-\alpha-\beta$
w	1	$s_\beta s_\alpha$	s_α	s_β	$s_\beta s_\alpha s_\beta$	$s_\alpha s_\beta$

.

3. (a) Let X, C be $n \times n$ matrices, with C invertible. Show that $e^{CXC^{-1}} = Ce^X C^{-1}$.

(b) Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. If $A \in G$, let $\mathrm{Ad}_A(X) = AXA^{-1}$. Show that $\mathrm{Ad}_A \in \mathfrak{gl}(\mathfrak{g})$; that is, show that $\mathrm{Ad}_A(X) \in \mathfrak{g}$ whenever $X \in \mathfrak{g}$, and $X \mapsto AXA^{-1}$ is a linear map.

(c) Since $\mathrm{ad}_X \in \mathfrak{gl}(\mathfrak{g})$, we can define $e^{\mathrm{ad}_X} \in \mathfrak{gl}(\mathfrak{g})$ by the matrix exponential. Prove that $e^{\mathrm{ad}_X} = \mathrm{Ad}_{e^X}$ for all $X \in \mathfrak{g}$.

Solution.

(a) Using the matrix exponential, we have

$$e^{CXC^{-1}} = \sum_{m \geq 0} \frac{(CXC^{-1})^m}{m!} = \sum_{m \geq 0} \frac{CX^mC^{-1}}{m!} = C\left(\sum_{m \geq 0} \frac{X^m}{m!}\right)C^{-1} = Ce^XC^{-1}.$$

(b) If $X \in \mathfrak{g}$, then by definition, $e^{tX} \in G$ for all $t \in \mathbb{R}$, so by part (a), for all $t \in \mathbb{R}$,

$$e^{t(AXA^{-1})} = Ae^{tX}A^{-1} \in G,$$

so again using the definition of the Lie algebra of G, $AXA^{-1} \in \mathfrak{g}$. Furthermore, Ad_A is linear in X since matrix multiplication is linear in X: $\mathrm{Ad}_A(aX + bY) = A(aX + bY)A^{-1} = aAXA^{-1} + bAYA^{-1} = a\mathrm{Ad}_A(X) + b\mathrm{Ad}_A(Y)$.

(c) Fix $X \in \mathfrak{g}$, and note that $e^X \in G$. We want to show that for all $Y \in \mathfrak{g}$, $e^{\mathrm{ad}_X}(Y) = \mathrm{Ad}_{e^X}(Y)$.
Method 1: Consider $F : \mathbb{R} \to \mathfrak{g}$ defined by $F(t) = e^{tX}Ye^{-tX} = \mathrm{Ad}_{e^{tX}}(Y)$, which is a smooth curve in the finite-dimensional vector space $M_n(\mathbb{R})$. Differentiating,

$$F'(t) = Xe^{tX}Ye^{-tX} - e^{tX}Ye^{-tX}X = [X, F(t)] = \mathrm{ad}_X(F(t)),$$

with initial condition $F(0) = Y$. This is a linear ODE with constant coefficients on the vector space $M_n(\mathbb{R})$, so by uniqueness of solutions, $F(t) = e^{t\,\mathrm{ad}_X}(Y)$. Setting $t = 1$ gives $\mathrm{Ad}_{e^X}(Y) = e^{\mathrm{ad}_X}(Y)$.
Method 2: We claim that $(\mathrm{ad}_X)^n(Y) = \sum_{j=0}^n \binom{n}{j}(-1)^{n-j}X^jYX^{n-j}$, which we prove by induction. For $n = 1$, $\mathrm{ad}_X(Y) = XY - YX$, which matches. For the inductive step, assuming the formula holds for n, we apply ad_X and use the identity $[X, AB] = [X, A]B + A[X, B]$ to obtain

$$(\mathrm{ad}_X)^{n+1}(Y) = \sum_{j=0}^{n} \binom{n}{j}(-1)^{n-j}\left(X^{j+1}YX^{n-j} - X^jYX^{n-j+1}\right),$$

and re-indexing and applying Pascal's identity $\binom{n}{j-1} + \binom{n}{j} = \binom{n+1}{j}$ yields the formula for $n + 1$. Using this, the degree-n term (in X) of $e^{\mathrm{ad}_X}(Y)$ is

$$\frac{(\mathrm{ad}_X)^n(Y)}{n!} = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{j!(n-j)!}X^jYX^{n-j},$$

which is exactly the degree-n term in the expansion of

$$e^X Y e^{-X} = \left(\sum_j \frac{X^j}{j!}\right) Y \left(\sum_k \frac{(-1)^k X^k}{k!}\right).$$

Summing over n gives $e^{\mathrm{ad}_X}(Y) = e^X Y e^{-X} = \mathrm{Ad}_{e^X}(Y)$.

4. Let G be a finite group, and equip $L^2(G) = \{f : G \to \mathbb{C}\}$ with the inner product $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$. By HW1 #4, $L^2(G)$ under left translation is the regular representation, which by Corollary 13 decomposes as $\bigoplus_\rho V_\rho^{\oplus d_\rho}$, where $d_\rho = \dim V_\rho$ and the sum runs over all irreducible representations. The goal of this problem is to find an explicit orthonormal basis realizing this decomposition, in analogy with the Peter–Weyl theorem.

For each irreducible representation (ρ, V_ρ) of G, fix a G-invariant inner product on V_ρ and an orthonormal basis $\{e_i\}_{i=1}^{d_\rho}$ with respect to that inner product. Let $\rho_{ij} : G \to \mathbb{C}$ be the matrix coefficient $\rho_{ij}(g) = \langle e_i, \rho(g) e_j \rangle$; these are the examples of matrix coefficients discussed in class, where $\rho_{ij}(g)$ is the (i,j)-entry of $\rho(g)$.

(a) *(Orthogonality of matrix coefficients.)* Let ρ, π be irreducible representations of G. Show that

$$\langle \rho_{ij}, \pi_{kl} \rangle = \begin{cases} 0 & \text{if } \rho \not\cong \pi, \\ \dfrac{\delta_{ik}\delta_{jl}}{d_\rho} & \text{if } \rho = \pi. \end{cases}$$

In particular, the matrix coefficients of non-isomorphic irreducibles are orthogonal, and those of a single irreducible are orthogonal to one another.

(Hint: Mirroring the averaging approach we have used a few times, given a linear map $T : V_\pi \to V_\rho$, set $\hat{T} = \frac{1}{|G|} \sum_{g \in G} \rho(g)\, T\, \pi(g)^{-1}$. Show that $\hat{T}$ is G-equivariant, then apply Schur's lemma. Choose T to be a well-chosen rank-one map to extract the desired inner products.)

(b) Using part (a) and Corollary 14, show that $\left\{ \sqrt{d_\rho}\, \rho_{ij} : \rho \text{ irreducible},\ 1 \le i, j \le d_\rho \right\}$ is an orthonormal basis for $L^2(G)$.

Solution.

(a) For any linear map $T : V_\pi \to V_\rho$, define the averaging operator

$$\hat{T} = \frac{1}{|G|} \sum_{g \in G} \rho(g)\, T\, \pi(g)^{-1}.$$

We claim that $\hat{T}$ is G-equivariant: for any $h \in G$, re-indexing $g \mapsto h^{-1}g$ and using $\pi(h^{-1}g)^{-1} = \pi(g)^{-1}\pi(h)$,

$$\rho(h)\hat{T} = \frac{1}{|G|} \sum_g \rho(hg)\, T\, \pi(g)^{-1} = \frac{1}{|G|} \sum_g \rho(g)\, T\, \pi(g)^{-1}\pi(h) = \hat{T}\, \pi(h).$$

This means $\hat{T} \in \mathrm{Hom}_G(V_\pi, V_\rho)$, so by Schur's lemma, $\hat{T} = 0$ if $\rho \not\cong \pi$. If $\rho = \pi$, then $\hat{T} = \lambda I$ for some scalar λ; taking the trace of both sides and using $\mathrm{tr}(\rho(g) T \rho(g)^{-1}) = \mathrm{tr}(T)$ gives $\lambda d_\rho = \mathrm{tr}(\hat{T}) = \mathrm{tr}(T)$, so $\hat{T} = \frac{\mathrm{tr}(T)}{d_\rho} I$.

Now take $T : V_\pi \to V_\rho$ to be the rank-one map $T(v) = \langle e_l, v\rangle e_j$, where $e_l \in V_\pi$ and $e_j \in V_\rho$ are basis vectors. Since the inner product on V_π is G-invariant, $\langle e_l, \pi(g)^{-1} e_k\rangle = \langle \pi(g) e_l, e_k\rangle = \overline{\pi_{kl}(g)}$, so $T(\pi(g)^{-1}e_k) = \overline{\pi_{kl}(g)}\, e_j$. Therefore,

$$\begin{aligned}
\langle e_i, \hat{T}(e_k)\rangle &= \frac{1}{|G|}\sum_{g\in G} \langle e_i, \rho(g)\, T(\pi(g)^{-1} e_k)\rangle \\
&= \frac{1}{|G|}\sum_{g\in G} \overline{\pi_{kl}(g)}\, \langle e_i, \rho(g) e_j\rangle \\
&= \frac{1}{|G|}\sum_{g\in G} \rho_{ij}(g)\, \overline{\pi_{kl}(g)} \\
&= \langle \rho_{ij}, \pi_{kl}\rangle.
\end{aligned}$$

When $\rho \not\cong \pi$, $\hat{T} = 0$ immediately gives $\langle \rho_{ij}, \pi_{kl}\rangle = 0$. When $\rho = \pi$, T is the matrix unit E_{jl}, so $\mathrm{tr}(T) = \delta_{jl}$, and therefore $\hat{T} = \frac{\delta_{jl}}{d_\rho} I$, giving $\langle \rho_{ij}, \rho_{kl}\rangle = \langle e_i, \frac{\delta_{jl}}{d_\rho} e_k\rangle = \frac{\delta_{ik}\delta_{jl}}{d_\rho}$.

(b) By part (a), $\langle \sqrt{d_\rho}\, \rho_{ij}, \sqrt{d_\pi}\, \pi_{kl}\rangle = \delta_{\rho\pi}\delta_{ik}\delta_{jl}$, so the collection is orthonormal. By Corollary 14, $\dim L^2(G) = |G| = \sum_\rho d_\rho^2$, so the collection contains exactly $|G|$ orthonormal vectors in a space of dimension $|G|$, and is therefore an orthonormal basis.

5. Give an example of a matrix Lie group G and a matrix X such that $e^X \in G$ but X is not in the Lie algebra $\mathfrak{g}$ of G.

Solution. We need to find a matrix X such that $e^X \in G$ but $e^{tX} \notin G$ for some $t \in \mathbb{R}$. One example is if $G = O(n) = \{g \in GL_n(\mathbb{C}) | gg^T = I\}$ is the orthogonal group. Let X be the scalar matrix $X = i\pi I$. Then $e^{tX} = e^{it\pi} I$. When $t = 1$, this is $e^{i\pi} I = -I \in O(n)$ since $(-I)(-I)^T = I$. On the other hand, for general t, we have $e^{it\pi} I\, (e^{it\pi} I)^T = e^{2it\pi} I$, which equals I if and only if $t \in \mathbb{Z}$, so $X \notin \mathfrak{g}$.