Math 506, Spring 2026 – Homework 3

Due: Wednesday, 25th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. All assertions require proof unless otherwise stated. Typesetting your homework using LaTeX is recommended. For this homework, unless otherwise stated all groups are finite and all representations are finite dimensional and complex.

Notation:

- χ^λ is the irreducible character of the Specht module S^λ
- C_λ is the conjugacy class consisting of permutations of cycle type λ
- w_λ is a representative of C_λ.
- S_λ is the Young subgroup $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots S_{\lambda_{\ell(\lambda)}}$.

1. In this problem, we compute the character table for S_n. Define the matrices $A = (A_{\lambda\mu})$ and $B = (B_{\lambda\mu})$ by

$$A_{\lambda\mu} = |S_\lambda \cap C_\mu|, \qquad B_{\lambda\mu} = |S_\mu| K_{\lambda\mu},$$

   where rows and columns are ordered by lexicographic order, largest first.

   (a) Explain why $M^\mu = \mathrm{Ind}_{S_\mu}^{S_n} V_{\mathrm{triv}}$ (*a sketch is fine for this part*).

   (b) Show that A and B are upper triangular and invertible.

   (c) Prove that $B(A^T)^{-1}$ is the character table of S_n.

   (d) Compute A, B, and the character table of S_n when $n = 4$.

2. Let M be the character table of a group G.

   (a) Use the orthogonality relations to give a formula (defined up to sign) for $\det M$.

   (b) For any $\lambda \vdash n$, let $m_i(\lambda)$ be the number of parts of size i. Using part (a) in the case $G = S_n$, and also using Problem 1, prove the remarkable fact that for all n,

$$\prod_{\lambda \vdash n} \prod_{1 \le i \le \ell(\lambda)} \lambda_i = \prod_{\lambda \vdash n} \prod_{i \ge 1} m_i(\lambda)!.$$

3. In this problem, we consider Specht modules over an arbitrary field F, as is covered in James' book. With respect to this field, M^λ and S^λ are defined the same way as over $\mathbb{C}$ (in particular, the coefficient of each tabloid appearing in a polytabloid is ± 1). The Submodule Theorem (Theorem 30) still holds: with respect to the S_n-invariant bilinear form (no longer necessarily an inner product), any submodule of M^μ either contains S^μ or is contained in $(S^\mu)^\perp$.

   (a) Prove that $S^\mu \cap (S^\mu)^\perp$ is a submodule, and furthermore it either equals S^μ or it is the unique maximal proper submodule of S^μ. Prove that the quotient module $S^\mu/(S^\mu \cap (S^\mu)^\perp)$ is zero or irreducible.

   (b) In the case $\mu = (3, 1)$, show that over any field either $S^\mu \cap (S^\mu)^\perp = \{0\}$ or $(S^\mu)^\perp \subseteq S^\mu$. Give a basis of the quotient module $S^\mu/(S^\mu \cap (S^\mu)^\perp)$.

4. [THIS PROBLEM IS INCORRECT. PLEASE DISREGARD IT.] For a Young tableau T of shape $\lambda \vdash n$ (with entries $1, \ldots, n$), recall the row stabilizer R_T and column stabilizer C_T. Define the following elements of $\mathbb{C}[S_n]$:

$$a_T = \sum_{w \in R_T} w, \qquad b_T = \sum_{w \in C_T} (-1)^w w, \qquad c_T = a_T b_T.$$

   c_T is the *Young symmetrizer* associated to T. (b_T is what we called κ_T in Definition 27).

   (a) Show that $a_T \sigma = a_T$ for $\sigma \in R_T$, that $b_T w = (-1)^w b_T$ for $w \in C_T$, and that $c_T\{T\} = |R_T| e_T$.

   (b) Define $\phi : \mathbb{C}[S_n]c_T \to S^\lambda$ by $\phi(y c_T) = y c_T\{T\}$. Show that ϕ is an isomorphism of S_n-modules, so that $\mathbb{C}[S_n]c_T \cong S^\lambda$.

5. The *Hecke algebra* $\mathcal{H} := \mathcal{H}_q(S_n)$ is the $\mathbb{C}$-algebra with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$T_i^2 = (q-1)T_i + q, \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad T_i T_j = T_j T_i \text{ when } |i-j| \geq 2.$$

   (a) When $q = 1$, show an isomorphism between $\mathcal{H}$ and the group algebra $\mathbb{C}[S_n]$. (*no need to explicitly show bijectivity, since this can be tricky*)

   (b) A representation of an algebra A is a $\mathbb{C}$-algebra homomorphism $A \to \mathrm{End}(V)$ for some vector space V (sending 1 to the identity matrix). Determine the one-dimensional representations of $\mathcal{H}$.

6. The *Durfee square* of a partition λ is the largest square that fits inside its diagram (i.e. λ has a Durfee square of side length $\geq s$ if and only if $\lambda_s \geq s$). Prove that if λ has a Durfee square of size s and $\ell(\mu) < s$, then $\chi^\lambda(w_\mu) = 0$.