

Math 506, Spring 2026 – Homework 3

Due: Wednesday, 25th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. All assertions require proof unless otherwise stated. Typesetting your homework using LaTeX is recommended. For this homework, unless otherwise stated all groups are finite and all representations are finite dimensional and complex.

Notation:

- χ^λ is the irreducible character of the Specht module S^λ
- C_λ is the conjugacy class consisting of permutations of cycle type λ
- w_λ is a representative of C_λ .
- S_λ is the Young subgroup $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_{\ell(\lambda)}}$.

1. In this problem, we compute the character table for S_n . Define the matrices $A = (A_{\lambda\mu})$ and $B = (B_{\lambda\mu})$ by

$$A_{\lambda\mu} = |S_\lambda \cap C_\mu|, \quad B_{\lambda\mu} = |S_\mu|K_{\lambda\mu},$$

where rows and columns are ordered by lexicographic order, largest first.

- (a) Explain why $M^\mu = \text{Ind}_{S_\mu}^{S_n} V_{\text{triv}}$ (a sketch is fine for this part).
- (b) Show that A and B are upper triangular and invertible.
- (c) Prove that $B(A^T)^{-1}$ is the character table of S_n .
- (d) Compute A, B , and the character table of S_n when $n = 4$.

Solution.

- (a) By Example(b) in Lecture 7, the induced representation $\text{Ind}_H^G V_{\text{triv}}$ is the permutation representation of H cosets inside G . In the present situation, the μ -tabloids give a set of S_μ -coset representatives since the row stabilizer of any μ -tableau is isomorphic to S_μ . So $\text{Ind}_{S_\mu}^{S_n} V_{\text{triv}}$ is the permutation representation on μ -tabloids, which is M^μ .

- (b) Since dominance order refines lexicographic order, for triangularity we just need to show that $A_{\lambda\mu} = B_{\lambda\mu} = 0$ unless $\lambda \geq \mu$. For B , this is a consequence of the fact that the Kostka numbers obey this inequality (the shape dominates the content of any semistandard tableau). For A , if some element of C_μ lives in S_λ , then the numbers from each of its cycles must live completely inside some part in λ , which is a stronger condition than dominance.

Invertibility follows because $|S_\lambda \cap C_\lambda| > 0$ since placing $1, \dots, n$ in the cycles in order gives such an element, and $K_{\lambda\lambda} = 1$ since the semistandard tableau of shape λ with each box in row i filled with entry i has content λ .

- (c) Let C be the character table of S_n . We need to show that $CA^T = B$, and expanding this out gives

$$\sum_{\mu} \chi^\lambda(w_\mu) |S_\lambda \cap C_\mu| = |S_\mu| K_{\lambda\mu}.$$

By Frobenius reciprocity and part (a),

$$\dim \text{Hom}_{S_n}(S^\lambda, M^\mu) = \dim \text{Hom}_{S_n}(S^\lambda, \text{Ind}_{S_\mu}^{S_n} V_{\text{triv}}) = \dim \text{Hom}_{S_\mu}(\text{Res}_{S_\mu}^{S_n} S^\lambda, V_{\text{triv}}).$$

On the other hand,

$$\sum_{\mu} \chi^\lambda(w_\mu) |S_\lambda \cap C_\mu| = \sum_{\mu} \chi^\lambda(w_\mu) \chi_{\text{triv}}(w_\mu) |S_\lambda \cap C_\mu| = \frac{1}{|S_\mu|} \dim \text{Hom}_{S_\mu}(\text{Res}_{S_\mu}^{S_n} S^\lambda, V_{\text{triv}}),$$

so the two sides are equal.

- (d) Lexicographic order on partitions of size 4 is: $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$. With this ordering, we have

$$A = \begin{bmatrix} 6 & 8 & 3 & 6 & 1 \\ 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 24 & 6 & 4 & 2 & 1 \\ 0 & 6 & 4 & 4 & 3 \\ 0 & 0 & 4 & 2 & 2 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$C = B(A^T)^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 & 3 \\ 0 & -1 & 2 & 0 & 2 \\ 1 & 0 & -1 & -1 & 3 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}.$$

2. Let M be the character table of a group G .

- (a) Use the orthogonality relations to give a formula (defined up to sign) for $\det M$.

- (b) For any $\lambda \vdash n$, let $m_i(\lambda)$ be the number of parts of size i . Using part (a) in the case $G = S_n$, and also using Problem 1, prove the remarkable fact that for all n ,

$$\prod_{\lambda \vdash n} \prod_{1 \leq i \leq \ell(\lambda)} \lambda_i = \prod_{\lambda \vdash n} \prod_{i \geq 1} m_i(\lambda)!.$$

Solution.

- (a) Let M^* be the conjugate transpose of M . The column orthogonality relations (Problem 2 of Homework 2) show that $N := M^*M$ is the diagonal matrix with entries $N_{ii} = |G|/|C_i|$, where C_1, \dots, C_k are the conjugacy classes of G . Thus, $\det N = \prod_i |G|/|C_i|$.

Let $D = \det M$. Transposes fix determinants, so $\det M^* = \det \overline{M} = \overline{D}$. Taking the complex conjugate of an irreducible character gives an irreducible character (proof: if it decomposes, take the complex conjugate again, and so the original character also decomposes), so \overline{M} is a permutation of the rows of M i.e. $\overline{D} = \pm D$, so D is real or purely imaginary. The former case occurs when the number of pairs m of conjugate irreducible characters is even, and the latter is when m is odd. Therefore we have

$$D = \pm i^m \sqrt{\prod_i \frac{|G|}{|C_i|}}.$$

- (b) Let $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i(\lambda)!$. We will prove

$$\prod_{\lambda \vdash n} \prod_{1 \leq i \leq \ell(\lambda)} \lambda_i^2 = \prod_{\lambda \vdash n} z_\lambda, \tag{1}$$

and dividing both sides by $\prod_{\lambda \vdash n} \prod_{1 \leq i \leq \ell(\lambda)} \lambda_i$ gives the desired equation.

We compute D^2 in two ways in the case $G = S_n$. First, we use part (a) of this problem, noting that by Corollary 38(d), D is real, so $D^2 > 0$. We need to compute the conjugacy class sizes. To choose an element of C_λ , we choose one of the $n!$ orderings of $1, \dots, n$ and place the integers in the cycles. Cyclic permutations of each cycle and rearranging cycles of the same size don't change the permutation, so

$$|C_\lambda| = \frac{n!}{z_\lambda}.$$

Plugging this into the formula in part (a), we get

$$D^2 = \prod_{\lambda \vdash n} \frac{|S_n|}{|C_\lambda|} = \prod_{\lambda \vdash n} \frac{n!}{n!/z_\lambda} = \prod_{\lambda \vdash n} z_\lambda,$$

the right side of (1).

On the other hand, with the A and B matrices from Problem 1, we have $D = \det B(A^T)^{-1} = \det B / \det A$. A and B are triangular, so their determinants are just the products of their diagonal entries. We have

$$A_{\lambda\lambda} = |S_\lambda \cap C_\lambda| = \frac{|S_\lambda|}{\lambda_1 \lambda_2 \cdots \lambda_{\ell(\lambda)}},$$

since the probability that a permutation in S_k is a k -cycle is $1/k$, and

$$B_{\lambda\lambda} = |S_\lambda| K_{\lambda\lambda} = |S_\lambda|.$$

Therefore,

$$D = \frac{\det B}{\det A} = \prod_{\lambda \vdash n} \prod_{1 \leq i \leq \ell(\lambda)} \lambda_i,$$

and hence D^2 equals the left side of (1).

3. In this problem, we consider Specht modules over an arbitrary field F , as is covered in James' book. With respect to this field, M^λ and S^λ are defined the same way as over \mathbb{C} (in particular, the coefficient of each tabloid appearing in a polytabloid is ± 1). The Submodule Theorem (Theorem 30) still holds: with respect to the S_n -invariant bilinear form (no longer necessarily an inner product), any submodule of M^μ either contains S^μ or is contained in $(S^\mu)^\perp$.

- (a) Prove that $S^\mu \cap (S^\mu)^\perp$ is a submodule, and furthermore it either equals S^μ or it is the unique maximal proper submodule of S^μ . Prove that the quotient module $S^\mu / (S^\mu \cap (S^\mu)^\perp)$ is zero or irreducible.
- (b) In the case $\mu = (3, 1)$, show that over any field either $S^\mu \cap (S^\mu)^\perp = \{0\}$ or $(S^\mu)^\perp \subseteq S^\mu$. Give a basis of the quotient module $S^\mu / (S^\mu \cap (S^\mu)^\perp)$.

Solution.

- (a) If $S^\mu \subseteq (S^\mu)^\perp$, then everything follows: $S^\mu \cap (S^\mu)^\perp = S^\mu$ is (trivially) a submodule of S^μ , and the quotient is zero. So assume that $S^\mu \cap (S^\mu)^\perp \subsetneq S^\mu$. First, we show that it is a submodule. S^μ itself is a submodule, and by S_n -invariance of the inner product, if $v \in (S^\mu)^\perp$, then for all $u \in S^\mu$ and all $\sigma \in S_n$,

$$\langle u, \sigma v \rangle = \langle \sigma^{-1} u, v \rangle = 0$$

since $\sigma^{-1} u \in S^\mu$; thus $\sigma v \in (S^\mu)^\perp$, so $(S^\mu)^\perp$ is a submodule of M^μ , and the intersection of submodules is a submodule.

Now, the Submodule Theorem says that any submodule U of M^μ either contains S^μ or is contained in $(S^\mu)^\perp$; if U is a proper submodule of S^μ , the former condition cannot hold, and so the latter one must, and we have $U \subseteq S^\mu \cap (S^\mu)^\perp$.

Finally, consider the quotient module $S^\mu/(S^\mu \cap (S^\mu)^\perp)$; by the fourth (or “lattice”) isomorphism theorem for modules, there is a bijection between submodules of S^μ containing $S^\mu \cap (S^\mu)^\perp$ and submodules of $S^\mu/(S^\mu \cap (S^\mu)^\perp)$. By the maximality proven in the previous paragraph, there are no proper submodules of S^μ properly containing $S^\mu \cap (S^\mu)^\perp$, so $S^\mu/(S^\mu \cap (S^\mu)^\perp)$ is irreducible.

(b) When $\mu = (3, 1)$, we have four tabloids:

$$\frac{\overline{1 \ 2 \ 3}}{\underline{4}}, \quad \frac{\overline{1 \ 2 \ 4}}{\underline{3}}, \quad \frac{\overline{1 \ 3 \ 4}}{\underline{2}}, \quad \frac{\overline{2 \ 3 \ 4}}{\underline{1}},$$

and M^μ is the span of these tabloids. Let m_i be the unique tabloid with i in the second row. The polytabloids are

$$e_{i,\ell} = \frac{\overline{i \ j \ k}}{\underline{\ell}} - \frac{\overline{\ell \ j \ k}}{\underline{i}} = -e_{\ell,i}.$$

for all choices i, ℓ . Taking $\ell > i < j < k$ makes the polytabloid standard, which as we’ve shown in class (and a direct check also shows) gives a basis for S^μ :

$$\frac{\overline{1 \ 2 \ 3}}{\underline{4}} - \frac{\overline{2 \ 3 \ 4}}{\underline{1}}, \quad \frac{\overline{1 \ 2 \ 4}}{\underline{3}} - \frac{\overline{2 \ 3 \ 4}}{\underline{1}}, \quad \frac{\overline{1 \ 3 \ 4}}{\underline{2}} - \frac{\overline{2 \ 3 \ 4}}{\underline{1}}.$$

Let $v \in (S^\mu)^\perp$. Then $\langle v, e_{i,\ell} \rangle = 0$ for all i, ℓ , so the coefficients of m_i and m_ℓ in v are the same for all i, ℓ . This implies that $v \in \text{span}(m_1 + m_2 + m_3 + m_4)$, and conversely any such vector is in $(S^\mu)^\perp$.

Now, if $\text{char } F \neq 2$, then $m_1 + m_2 + m_3 + m_4 \notin S^\mu$ since the sum of the coefficients is nonzero, while it is zero for any element of S^μ . Then $S^\mu \cap (S^\mu)^\perp = \{0\}$, and the Submodule Theorem says that S^μ is irreducible. The basis is the one given in class.

On the other hand, if $\text{char } F = 2$, then $m_1 + m_2 + m_3 + m_4 = m_1 - m_2 + m_3 - m_4 \in S^\mu$, so $(S^\mu)^\perp \subseteq S^\mu$. In this case, $S^\mu/(S^\mu)^\perp$ has dimension 2, and basis

$$\frac{\overline{1 \ 2 \ 3}}{\underline{4}} + \frac{\overline{2 \ 3 \ 4}}{\underline{1}}, \quad \frac{\overline{1 \ 2 \ 4}}{\underline{3}} + \frac{\overline{2 \ 3 \ 4}}{\underline{1}}.$$

In the quotient, the third basis vector of S^μ is the sum of the other two.

4. [\[THIS PROBLEM IS INCORRECT. PLEASE DISREGARD IT.\]](#)

For a Young tableau T of shape $\lambda \vdash n$ (with entries $1, \dots, n$), recall the row stabilizer R_T and column stabilizer C_T . Define the following elements of $\mathbb{C}[S_n]$:

$$a_T = \sum_{w \in R_T} w, \quad b_T = \sum_{w \in C_T} (-1)^w w, \quad c_T = a_T b_T.$$

c_T is the *Young symmetrizer* associated to T . (b_T is what we called κ_T in Definition 27).

- (a) Show that $a_T\sigma = a_T$ for $\sigma \in R_T$, that $b_Tw = (-1)^w b_T$ for $w \in C_T$, and that $c_T\{T\} = |R_T|e_T$.
- (b) Define $\phi : \mathbb{C}[S_n]_{c_T} \rightarrow S^\lambda$ by $\phi(yc_T) = y\{T\}$. Show that ϕ is an isomorphism of S_n -modules, so that $\mathbb{C}[S_n]_{c_T} \cong S^\lambda$.
5. The Hecke algebra $\mathcal{H} := \mathcal{H}_q(S_n)$ is the \mathbb{C} -algebra with generators T_1, T_2, \dots, T_{n-1} and relations

$$T_i^2 = (q-1)T_i + q, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \text{ when } |i-j| \geq 2.$$

- (a) When $q = 1$, show an isomorphism between \mathcal{H} and the group algebra $\mathbb{C}[S_n]$. (*no need to explicitly show bijectivity, since this can be tricky*)
- (b) A representation of an algebra A is a \mathbb{C} -algebra homomorphism $A \rightarrow \text{End}(V)$ for some vector space V (sending 1 to the identity matrix). Determine the one-dimensional representations of \mathcal{H} .

Solution.

- (a) Let s_i be the 2-cycle $(i, i+1)$. These permutations generate the symmetric group S_n and satisfy the relations

$$s_i^2 = 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad s_i s_j = s_j s_i \text{ when } |i-j| \geq 2.$$

When $q = 1$, the T_i 's satisfy the same relations, so the map $T_i \rightarrow s_i$ is an isomorphism $\mathcal{H}_1(S_n) \rightarrow \mathbb{C}[S_n]$.

- (b) If V is one-dimensional, every T_i acts by a scalar $t_i \in \mathbb{C}$. In the representation, the quadratic relation $T_i^2 = (q-1)T_i + q$ becomes $t_i^2 = (q-1)t_i + q$, or $(t_i - q)(t_i + 1) = 0$, so we must have $t_i = q$ or -1 . The braid relation $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ becomes $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$, and dividing both sides by $t_i t_{i+1}$ (which is nonzero if $q \neq 0$) gives $t_i = t_{i+1}$. Therefore, \mathcal{H} has two one-dimensional representations: one where every T_i acts by multiplication by q and the other where every T_i acts by multiplication by -1 . These are the q -analogues of the trivial and sign representations of S_n .

(Note: in the case $q = 0$, we have $t_i = 0$ or -1 for all i , and the braid relation argument no longer says all the t_i are equal. So in this highly degenerate case, we have 2^{n-1} one-dimensional representations.)

6. The Durfee square of a partition λ is the largest square that fits inside its diagram (i.e. λ has a Durfee square of side length $\geq s$ if and only if $\lambda_s \geq s$). Prove that if λ has a Durfee square of size s and $\ell(\mu) < s$, then $\chi^\lambda(w_\mu) = 0$.

Solution. This is a direct consequence of the Murnaghan–Nakayama rule. Let

$$\emptyset \subseteq \lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \dots \subseteq \lambda^{(k)} = \lambda$$

be a nested sequence of partitions such that $\lambda^{(i)} \setminus \lambda^{(i-1)}$ is a border-strip of $\lambda^{(i)}$ for each i . A border-strip can contain at most one box on the main diagonal, so $k \geq s$. The Murnaghan–Nakayama rule writes $\chi^\lambda(w_\mu)$ as a sum over such chains, where $k = \ell(\mu)$. Since $\ell(\mu) < s$, there are no such chains, and $\chi^\lambda(w_\mu) = 0$.