Math 506, Spring 2026 – Homework 2

Due: Wednesday, February 25th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. All assertions require proof unless otherwise stated. Typesetting your homework using LaTeX is recommended.

For this homework, unless otherwise stated all groups are finite and all representations are finite dimensional and complex.

1. Let V and W be G-representations, with W irreducible. Prove that the following map $V \to V$ is a G-equivariant projection onto the W-isotypic component of V:

$$\pi_W := \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot g.$$

(*In the case where W is the trivial representation, this is Proposition 9.*)

Solution. To show that π_W is G-equivariant, we compute

$$h\pi_W = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot hg = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(h^{-1}gh)} \cdot gh = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot gh = \pi_W h,$$

where in the second equality, we substitute $g \mapsto hgh^{-1}$, and the third equality is because χ_W is a class function.

Since V is a direct sum of irreducibles, every vector in V is a linear combination of vectors living in irreducible direct summands. Let U be an irreducible subrepresentation of V, and let $u \in U$. To complete the proof we will show that if $U \cong W$, then $\pi_W(u) = u$, and otherwise, $\pi_W(u) = 0$.

Note first that U being a subrepresentation means that $g \cdot U \subseteq U$ for all $g \in G$, so $\pi_W \cdot U \subseteq U$. This means that $\pi_W|U \in \operatorname{Hom}_G(U, U)$, so by Schur's Lemma, $\pi_W|U = \lambda \cdot I$ for some $\lambda \in \mathbb{C}$. We have

$$\lambda = \frac{\operatorname{Tr}(\pi_W|U)}{\dim U} = \frac{\dim W}{\dim U \cdot |G|} \sum_{g \in G} \overline{\chi_W(g)} \chi_U(g) = \frac{\dim W}{\dim U} (\chi_W, \chi_U) = \begin{cases} 1, & \text{if } U \cong W, \\ 0, & \text{otherwise,} \end{cases}$$

proving the claim.

2. Prove the last statement in Corollary 16. That is, given that the irreducible characters form an orthonormal basis of $\mathbb{C}_{cl}(G)$ with respect to the Hermitian inner product $(\cdot,\cdot)$, show that

$$\sum_{\chi\text{:irred}} \overline{\chi(g)}\chi(h) = \begin{cases} |C(g)|, & \text{if } g \sim h, \\ 0, & \text{otherwise,} \end{cases}$$

where $C(g)$ is the centralizer of g in G.

(*Hint: use properties of unitary matrices.*)

Solution. The key linear algebra result is that a square matrix has (Hermitian) orthonormal rows if and only if it has (Hermitian) orthonormal columns; such a matrix is called *unitary*. Let A^* denote the conjugate transpose of A; the statement that the rows are orthonormal is equivalent to the equation $AA^* = I$. But this means that A and A^* are inverse matrices, so we can equally write $A^*A = I$, which is the statement that the columns are orthonormal. (Throughout, we can be loose about which factor in the inner product has the complex conjugate since $\overline{I} = I$.)

We need to fit the character table into this story. Let $\chi_1, \dots, \chi_n$ be the irreducible characters of G, and let $g_1, \dots, g_n$ be a set of conjugacy class representatives. Recall that $|G|/|C(g)|$ equals the size of the conjugacy class of g. Let A be the matrix (A_{ij}) where

$$A_{ij} = \frac{1}{|C(g_j)|^{1/2}}\,\chi_i(g_j).$$

Then character orthogonality (Theorem 10) tells us that A is a unitary matrix: $AA^* = I$. The reverse equality, $A^*A = I$ then expands as

$$\sum_i \frac{1}{|C(g_j)|^{1/2}|C(g_k)|^{1/2}}\,\overline{\chi_i(g_j)}\chi_i(g_k) = \delta_{jk}.$$

Scaling this formula (in the case where $j = k$), we have

$$\sum_i \overline{\chi_i(g_j)}\chi_i(g_k) = \begin{cases} |C(g_j)|, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

3. Let G and H be finite groups. If V is an irreducible representation of G and W is an irreducible representation of H, show that $V \otimes W$ is an irreducible representation of $G \times H$ (i.e. define an action of $G \times H$ on $V \otimes W$ and show that the resulting representation is irreducible). (This is called the *external tensor product*; don't confuse it with the *internal tensor product* we've already looked it.) Show that every irreducible representation of $G \times H$ is of this form.

(*Hint: use character theory.*)

Solution. Let $G \times H$ act on $V \otimes W$ via $(g,h)\cdot(v \otimes w) := gv \otimes hw$. It can be checked directly that this is a linear group action.

For the rest, we use character theory. If G has k conjugacy classes, and H has ℓ conjugacy classes, then $G \times H$ has $k\ell$ conjugacy classes, and the same goes for irreducible representations. The action defined above means that the matrix for (g,h) on $V \otimes W$ is the Kronecker product of the matrices for g (on V) and h (on W); thus $\chi_{V\otimes W}(g,h) = \chi_V(g)\chi_W(h)$. We have

$$\begin{aligned}
(\chi_{V\otimes W}, \chi_{V'\otimes W'}) &= \frac{1}{|G\times H|}\sum_{\substack{g\in G\\ h\in H}} \overline{\chi_{V\otimes W}(g,h)}\chi_{V'\otimes W'}(g,h) \\
&= \left(\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_{V'}(g)\right)\left(\frac{1}{|H|}\sum_{h\in H}\overline{\chi_W(h)}\chi_{W'}(h)\right) \\
&= (\chi_V, \chi_{V'})(\chi_W, \chi_{W'}) \\
&= \begin{cases} 1, & \text{if } V\cong V', W\cong W' \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}$$

so $V \otimes W$ is irreducible. Furthermore, ranging over all k irreducible G-representations and all ℓ irreducible H-representations, we have constructed $k\ell$ pairwise-nonisomorphic irreducible $(G\times H)$-representations, hence all of them.

4. Prove *directly* that the induced representation given in Definition 17b is unique up to isomorphism. That is, fix two sets of coset representatives $\sigma_1, \dots, \sigma_k$ and $\sigma'_1, \dots, \sigma'_k$ for G/H with $\sigma_i^{-1}\sigma'_i \in H$. Let

$$V = \bigoplus_{i=1}^{k} W_i, \qquad W_i = \{w_i | w \in W\}, \qquad \text{with action } g\cdot w_i := (h\cdot w)_j, \text{ where } g\sigma_i = \sigma_j h,$$

$$V' = \bigoplus_{i=1}^{k} W_i, \qquad W'_i = \{w'_i | w \in W\}, \qquad \text{with action } g\cdot w'_i := (h'\cdot w)_j, \text{ where } g\sigma'_i = \sigma'_j h'.$$

Give an explicit G-isomorphism $V \to V'$.

Solution. For each i, we have $\eta_i := (\sigma'_i)^{-1}\sigma_i \in H$. Consider the map $V \to V$ given by

$$w_j \mapsto (\eta_j w)'_j,$$

which is bijective on each summand $W_j \to W'_j$.

If $g\sigma_i = \sigma_j h$, we have $g\sigma'_i = g\sigma_i\eta_i^{-1} = \sigma_j h\eta_i^{-1} = \sigma'_j\eta_j h\eta_i^{-1}$, so

$$g\cdot w_i = (h\cdot w)_j \mapsto (\eta_j h\cdot w)'_j = g\cdot(\eta_i w)'_i.$$

Therefore, the map is G-equivariant, and since it is bijective, it is an isomorphism.

5. There is another definition of induced representation that is more natural in some settings. Fix an H-representation (ρ, W). Let $\mathrm{Ind}_H^G \rho$ refer to the definition from class of the induced representation, corresponding to a fixed set $\sigma_1, \ldots, \sigma_k$ of (left) coset representatives for G/H. Let

$$\mathrm{ind}_H^G \rho = \{ f : G \to W | f(hg) = \rho(h) f(g) \text{ for all } h \in H, g \in G \},$$

with G-action given by

$$g \cdot f(g') := f(g'g).$$

(a) Prove that $\mathrm{ind}_H^G \rho$ is a G-representation. Specifically, prove that $g \cdot f(g') := f(g'g)$ is a G-action on the space of functions $G \to W$, and that if $f \in \mathrm{ind}_H^G \rho$, then so is $g \cdot f$.

(b) Prove that $\mathrm{Ind}_H^G \rho \cong \mathrm{ind}_H^G \rho$ by showing that

$$w_i \mapsto f_{w,i}, \qquad \text{where} \qquad \underset{(h \in H)}{f_{w,i}(h\sigma_j^{-1})} = \begin{cases} h \cdot w, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

is a G-isomorphism.

Solution.

(a) We have

$$(g \cdot (h \cdot f))(g') = (h \cdot f))(g'g) = f(g'gh) = ((gh) \cdot f)(g'),$$

so this is a well-defined G-action on the space of functions $G \to W$. Note also that the action is linear: $g \cdot (af + bf') = a(g \cdot f) + b(g \cdot f')$.
If $f \in \mathrm{ind}_H^G \rho$, let $f' = g' \cdot f$. Then, if $h \in H, g \in G$, we have

$$f'(hg) = (g' \cdot f)(hg) = f(hgg') = \rho(h) f(gg') = \rho(h)(g' \cdot f)(g) = \rho(h) f'(g),$$

where the third equality is because $f \in \mathrm{ind}_H^G \rho$. Then also $f' \in \mathrm{ind}_H^G \rho$.

(b) The function $f_{w,i}$ is a well-defined function $G \to W$ since every element of G has a unique factorization $g = \sigma_j h^{-1}$, $h \in H$, and so $g^{-1} = h\sigma_j^{-1}$. It is left-H-equivariant since if $h' \in H$ $f_{w,i}(h'g) = f_i(h'h\sigma_j) = h' \cdot f_{w,i}(h\sigma_j^{-1})$. The map $w_i \mapsto f_{w,i}$ is injective since it is injective on every W_i, and surjective since its inverse map is

$$f \mapsto \bigoplus_{i=1}^{k} (f(\sigma_i^{-1}))_i.$$

Finally, we show that the map $w_i \mapsto f_{w,i}$ is G-equivariant. If $g\sigma_i = \sigma_j h, h \in H$, then $g \cdot w_i = (h \cdot w)_j \mapsto f_{h \cdot w, j}$. On the other hand, if $g\sigma_{m'} = \sigma_m h'$, then $\sigma_m^{-1} g = h'\sigma_{m'}^{-1}$, and

$$\begin{aligned}
(g \cdot f_{w,i})(h''\sigma_m^{-1}) &= f_{w,i}(h''\sigma_m^{-1} g) \\
&= f_{w,i}(h''h'\sigma_{m'}^{-1}) \\
&= \begin{cases} h''h' \cdot w, & \text{if } i = m', \\ 0, & \text{otherwise.} \end{cases} \\
&= \begin{cases} h''h \cdot w, & \text{if } j = m, \\ 0, & \text{otherwise.} \end{cases} \\
&= f_{h \cdot w, j}(h''\sigma_m^{-1}),
\end{aligned}$$

where the fourth inequality is because if $i = m'$ then we must have $j = m$ and $h = h'$. Thus the map is G-equivariant and hence an isomorphism.

6. A *virtual representation* over $\mathbb{C}$ of a finite group G is a formal integer linear combination of irreducible G-representations (if all coefficients are nonnegative, this is an honest G-representation). Likewise, a *virtual character* is an element of $\mathbb{C}_{\mathrm{cl}}(G)$ with integer coefficients in the basis of irreducible characters.

   (a) If χ is a virtual character and $(\chi, \chi) = 1$, prove that (exactly) one of $\pm\chi$ is an irreducible character.

   (b) In the language of Lecture 9, fix a one-dimensional representation $\nu : L^\times \to \mathbb{C}$, let $\alpha = \nu|_{K^\times}$, and let ρ_ν be the virtual representation

$$\rho_\nu := \widetilde{\rho}_1 \otimes \rho_{\alpha,1} - \rho_{\alpha,1} - \mathrm{Ind}_{L^\times}^G \nu.$$

   Compute the virtual character of ρ_ν, and prove that it is irreducible.

Solution.

(a) Since the irreducible G-characters form an orthonormal basis of $\mathbb{C}_{\mathrm{cl}}(G)$, χ has a unique decomposition

$$\chi = \sum_i a_i \chi_i$$

as a linear combination into irreducible characters, and since χ is a virtual character, the a_i are all integers. By orthonormality, we have

$$1 = (\chi, \chi) = \sum_{i,j} (\chi_i, \chi_j) = \sum_i a_i^2,$$

and so one $a_j = \pm 1$ and the rest are 0. If $a_j = 1$, then $\chi = \chi_j$ is an irreducible character, and if $a_j = -1$, then $-\chi$ is. It is easy to see which; just look at $\chi(1)$, which is positive for any honest character.

(b) As in the lecture notes, we will render $GL_2(\mathbb{F})_q$ characters via their values on the conjugacy classes $a_x, b_x, c_{x,y}, d_{x,y}$. The results are in the following table, with explanation below.

	a_x	b_x	$c_{x,y}$	$d_{x,y}$
$\widetilde{\chi}_1$	q	0	1	1
$\chi_{\alpha,1}$	$(q+1)\alpha(x)$	$\alpha(x)$	$\alpha(x)+\alpha(y)$	0
$\widetilde{\chi}_1 \otimes \chi_{\alpha,1}$	$q(q+1)\alpha(x)$	0	$\alpha(x)+\alpha(y)$	0
$\mathrm{Ind}_{L^\times}^G \nu$	$q(q-1)\nu(x)$	0	0	$\nu(x+y\sqrt{D})+\overline{\nu}(x-y\sqrt{D})$
ξ_ν	$(q-1)\nu(x)$	$-\nu(x)$	0	$-\nu(x+y\sqrt{D})+-\overline{\nu}(x-y\sqrt{D})$

$\widetilde{\chi}_1$ is given in the lecture notes (setting α to the trivial representation $K\times \to \mathbb{C}$). Similarly, $\chi_{\alpha,1}$ is given in the lecture notes (setting $\beta = 1$). To take the tensor product, we just multiply the characters pointwise, obtaining the third line of the table. To compute the induced character in the next line, apply Proposition 18. Finally, the virtual character is obtained by pointwise addition/subtraction, noting for the first two columns that if $x \in K^\times, \nu(x) = \alpha(x)$.

For irreducibility, since $\nu(1) = 1$, $\chi_\nu(1) = q-1 > 0$, so by part (a) if $(\chi_\nu, \chi_\nu) = 1$, then ρ_ν is an irreducible representation, and a direct computation shows that indeed this is the case.

(I know that both the induced character and (χ_ν, χ_ν) are somewhat long and technical. I'm happy to discuss these computations in person. Answers without the full details for those computations are fine, although I believe it's invaluable to get your hands dirty with this sort of thing from time to time.)

7. Again use the setting of Lecture 9, and recall the Weil group $W := L^\times \rtimes \langle\sigma\rangle$, where $\sigma^2 = 1$ and $\sigma\ell = \overline{\ell}\sigma$ (σ can be thought of as the nontrivial automorphism of L fixing K).

Let τ be a two-dimensional representation of W, and write $\tau|_{L^\times} = \nu_1 \oplus \nu_2$.

(a) If τ is irreducible, show that $\overline{\nu_1} = \nu_2$, and both ν_1 and ν_2 do not factor through the norm map $N_{L/K}$.

(b) If τ is reducible, show that ν_1 and ν_2 factor through $N_{L/K}$.

Solution.

(a) Let $\nu := \nu_1$, and let v be a vector on which $L^\times$ acts as ν i.e. such that $\tau(\ell)v = \nu(\ell)v$ for all $\ell \in L^\times$. Since τ is irreducible, $\tau(\sigma)v$ must be linearly independent of v; let $w := \tau(\sigma)v$. Then, using the semi-direct product relation in W,

$$\tau(\ell)w = \tau(\ell)\tau(\sigma)v = \tau(\ell\sigma)v = \tau(\sigma\overline{\ell})v = \tau(\sigma)\tau(\overline{\ell})v = \tau(\sigma)\overline{\nu}(\ell)v = \overline{\nu}(\ell)\tau(\sigma)v = \overline{\nu}(\ell)w,$$

where the penultimate equality is because $\overline{\nu}(\ell)$ is a scalar. Thus, $\tau|_{L^\times} = \nu \oplus \overline{\nu}$. Furthermore, if ν factored through the norm map, we would have $\nu(\ell) = \nu(N_{L/K}(\ell)) = \nu(N_{L/K}(\overline{\ell})) = \nu(\overline{\ell})$, so $\nu = \overline{\nu}$. Therefore, $\tau(\ell)$ is just the scalar matrix

$$\begin{bmatrix} \nu(\ell) & \\ & \nu(\ell) \end{bmatrix},$$

Let v be an eigenvector for the matrix $\tau(\sigma)$. The span of v is invariant under σ, and also under $L^\times$ since $\tau(\ell)v = \nu(\ell)v$, so this span forms a W-invariant subspace, contradicting irreducibility.

(b) We will focus on one of the subrepresentations, and the other is similar. Since τ is reducible, let v be a vector whose span is W-invariant. Define ν as the map $L^\times \to \mathbb{C}$ given by $\tau(\ell)v = \nu(\ell)v$. v must also be an eigenvector for σ: $\tau(\sigma)v = \lambda v$ for some $\lambda \in \mathbb{C}$. Then we have

$$\tau(\sigma\ell)v = \tau(\overline{\ell}\sigma)v = \tau(\overline{\ell})\tau(\sigma)v = \tau(\overline{\ell})\lambda v = \lambda\tau(\overline{\ell})v = \lambda\nu(\overline{\ell})v,$$

and on the other hand

$$\tau(\sigma\ell)v = \tau(\sigma)\tau(\ell)v = \tau(\sigma)\nu(\ell)v = \nu(\ell)\tau(\sigma)v = \nu(\ell)\lambda v,$$

so $\nu(\ell) = \nu(\overline{\ell})$ for all ℓ, meaning that ν factors through the norm map.