Math 506, Spring 2026 – Homework 1

Due: Wednesday, February 11th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. All assertions require proof unless otherwise stated. Typesetting your homework using LaTeX is recommended.

For this homework, unless otherwise stated all groups are finite and all representations are finite dimensional and complex.

1. Let V and W be G-representations.

   (a) In Lecture 2, we defined G-actions on duals and tensor products; combining these gives $V^* \otimes W$ the structure of a G-representation. Under the (vector space) identification
   $$V^* \otimes W \cong \mathrm{Hom}(V, W) \qquad\qquad (\varphi \otimes w)(v) := \varphi(v)w,$$
   we obtain a G-action on $\mathrm{Hom}(V, W)$. Show that this is the same representation that we defined on $\mathrm{Hom}(V, W)$ in Lecture 2.

   (b) Show that $\phi \in \mathrm{Hom}(V, W)$ lies in the isotypic component of the trivial representation if and only if it is G-equivariant.

   Solution.

   (a) The contragredient representation V^* has the action $g\varphi(v) = \phi(g^{-1}v)$. Then the tensor product $V^* \otimes W$ has the action
   $$g(\varphi \otimes w)(v) = (g\varphi \otimes gw)(v) = \varphi(g^{-1}v) \otimes gw.$$
   Under the identification $(\varphi \otimes w)(v) := \varphi(v)w$, let $\phi \in \mathrm{Hom}(V, W)$ be associated to the pure tensor $\varphi \otimes w$. Then since $\varphi(v)$ is a scalar, under the Lecture 2 action $g\phi$ is the map
   $$(g\phi)(v) = g\phi(g^{-1}v) = g(\varphi(g^{-1}v)w) = \varphi(g^{-1}v)gw = (g\varphi)(v)gw.$$
   From the other side,
   $$g(\varphi \otimes w)(v) = (g\varphi \otimes gw)(v) \leftrightarrow (g\varphi)(v)gw,$$
   so the actions are the same.

(b) Unpacking the wording, we need to show that $g\phi = \phi$ for all $g \in G$ if and only if $\phi \in \mathrm{Hom}_G(V, W)$. If $\phi \in \mathrm{Hom}_G(V, W)$, then ϕ commutes with the action of the group, so $(g\phi)(v) = g\phi(g^{-1}v) = gg^{-1}\phi(v) = \phi(v)$, and ϕ is fixed by the action of G. Conversely, if $(g\phi)(v) = \phi(v)$ for all $v \in V, g \in G$, then we have $g^{-1}\phi(gv) = \phi(v)$ for all $g \in G, v \in V$, and acting by g on both sides, $\phi(gv) = g\phi(v)$, meaning $\phi \in \mathrm{Hom}_G(V, W)$.

2. The *permutation representation* V_{perm} of S_3 is its permutation action on $\mathbb{C}^3$:

$$w \cdot e_i := e_{w(i)}, \qquad i = 1, 2, 3.$$

Without using character theory, show that $V_{\mathrm{perm}} \cong V_{\mathrm{triv}} \oplus V_{\mathrm{ref}}$. That is, find a one-dimensional fixed subspace (i.e. a copy of the trivial representation), and a complementary two-dimensional subspace that is S_3-invariant and irreducible (and therefore is the reflection representation, which you can assume is the only two-dimensional irreducible representation of S_3.)

Solution. The vector $e_1+e_2+e_3$ is fixed, so spans a copy of the trivial representation. $W = \mathrm{span}(e_1 - e_2, e_2 - e_3)$ is a complementary subspace, and it is S_3-invariant since (12) and (23) generate S_3 and

$$(12)\cdot(e_1-e_2) = -(e_1-e_2) \in W, \qquad (12)\cdot(e_2-e_3) = e_1-e_3 = (e_1-e_2)+(e_2-e_3) \in W,$$

$$(23)\cdot(e_1-e_2) = e_1-e_3 = (e_1-e_2)+(e_2-e_3) \in W, \qquad (23)\cdot(e_2-e_3) = -(e_2-e_3) \in W.$$

If W isn't irreducible, it has a proper nontrivial S_3-invariant subspace, which must be one-dimensional, and therefore spanned by a mutual eigenvector of every element of S_3 However, this isn't the case since

$$(12) \cdot [a(e_1 - e_2) + b(e_2 - e_3)] = (b - a)(e_1 - e_2) + b(e_2 - e_3),$$

$$(23) \cdot [a(e_1 - e_2) + b(e_2 - e_3)] = a(e_1 - e_2) + (a - b)(e_2 - e_3).$$

If $a(e_1 - e_2) + b(e_2 - e_3)$ is an eigenvector of (12), then $b = 0$, and if it's an eigenvector of (23), then $a = 0$, so the only mutual eigenvector is 0, and W is irreducible.

3. We generalize the setting of the previous problem. Let X be a finite set with a G-action. The *permutation representation* associated to this action is the vector space $V = V_X = \{v_x | x \in X\}$, and the action $gv_x := v_{g\cdot x}$ turns V into a representation.

(a) Show that $\chi_V(g)$ is the number of elements of X fixed by g.

(b) Show that the number of (distinct) G-orbits in X equals the multiplicity of the trivial representation inside V.

Solution.

(a) The action of g permutes the elements of X, so the matrix $\rho_V(g)$ is a permutation matrix. Its trace is the number of 1's on the diagonal, i.e. the number of fixed points of the action of g on X.

(b) If $Y \subseteq X$ is a G-orbit, then V_Y is a subrepresentation of V_X, so we need only show that if G acts transitively on X, V_X has exactly one copy of the trivial representation. Now, $\sum_{x\in X} v_x$ is fixed by G, so spans a copy of the trivial representation. Conversely, suppose that

$$v = \sum_{x\in X} a_x v_x$$

is an eigenvector for each $g \in G$. The action of g preserves $\sum a_x$, so in fact v must be fixed by all $g \in G$. Then since the action of G is transitive, for any $x, y \in X$ there exists $g \in G$ such that $gx = y$. But then in $gv = \sum_{x\in X} a_x v_{gx}$, the coefficient of v_y is a_x, so we must have $a_x = a_y$ for all $x, y \in X$, and v is a multiple of $\sum_{x\in X} v_x$.

4. Let G be a finite group. Prove that the following representations are isomorphic:

$$R_1 = \langle v_g | g \in G\rangle, \qquad g \cdot v_h := v_{gh};$$

$$\begin{aligned} R_2 &= \langle v_g | g \in G\rangle, & g \cdot v_h &:= v_{hg^{-1}}; \\ R_3 &= \langle f : G \to \mathbb{C}\rangle, & g \cdot f(h) &:= f(g^{-1}h); \\ R_4 &= \langle f : G \to \mathbb{C}\rangle, & g \cdot f(h) &:= f(hg). \end{aligned}$$

(Any of these is called the *regular representation* of G. R_1 is the permutation representation associated to the left action of G on itself, while R_2 is the permutation representation associated to the right action of G on itself. R_1 and R_3 are each called the *left regular representation*, while R_2 and R_4 are called the *right regular representation*. The definitions in terms of functions tend to be more useful when working with infinite groups.)

Solution. For the isomorphism $R_1 \to R_2$, send $v_h \mapsto v_{h^{-1}}$. Then

$$R_1(g)v_h = v_{gh} \mapsto v_{h^{-1}g^{-1}} = R_2(g)v_{h^{-1}}.$$

Similarly, the map $f(h) \mapsto f(h^{-1})$ is an isomorphism $R_3 \to R_4$ since

$$[R_3(g)f](h) = f(g^{-1}h) \mapsto f(h^{-1}g) = [R_4(g)f](h^{-1}).$$

Finally, the map $R_1 \to R_3$ defined by sending v_h to the characteristic function ch_h for h is an isomorphism since $g\mathrm{ch}_h(k) = \mathrm{ch}_h(g^{-1}k) = \mathrm{ch}_{gh}(k)$, so

$$R_1(g)v_h = v_{gh} \mapsto \mathrm{ch}_{gh} = R_3(g)\mathrm{ch}_h.$$

5. Let V be an irreducible representation of the finite group G. To prove Maschke's Theorem, we constructed a G-invariant Hermitian inner product. Show that this inner product is unique up to scalar multiple. (*Hint: Hermitian inner products on* $\mathbb{C}^n$ *are classified by* $n \times n$ *positive-definite Hermitian matrices.*)

   Solution. Let $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_1$ be two G-invariant Hermitian inner products. There exist $n \times n$ positive-definite Hermitian matrices M and N such that

$$\langle v, w \rangle_1 = \overline{w}^T M v \qquad \text{and} \qquad \langle v, w \rangle_2 = \overline{w}^T N v.$$

   Since positive-definite matrices are invertible, we can write

$$\langle v, w \rangle_1 = \overline{w}^T N N^{-1} M v = \langle P v, w \rangle_2$$

   where $P = N^{-1}M$. Since both inner products are G-invariant, we have

$$\langle Pv, w \rangle_2 = \langle v, w \rangle_1 = \langle gv, gw \rangle_1 = \langle Pgv, gw \rangle_2 = \langle g^{-1}Pgv, w \rangle_2$$

   for all vectors $v, w \in V$, and all $g \in G$. This means that $g^{-1}Pg = P$ for all $g \in G$, so $P \in \mathrm{End}_G(V)$. Therefore, by Schur's Lemma, since V is irreducible, P is a scalar matrix, so the inner products differ only by a scalar multiple.

6. Show that for any representation V

$$\chi_{\mathrm{Sym}^2 V} = \frac{1}{2}[\chi_V(g)^2 + \chi_V(g^2)] \qquad \text{and} \qquad \chi_{\bigwedge^2 V} = \frac{1}{2}[\chi_V(g)^2 - \chi_V(g^2)].$$

   Use this to show that

$$V \otimes V \cong \mathrm{Sym}^2 V \oplus \bigwedge^2 V.$$

   Solution. Let $g \in G$, and let $\lambda_1, \dots, \lambda_k$ be the eigenvalues for the action of g on V. Then the eigenvalues for the action of g on $\mathrm{Sym}V$ are $\{\lambda_i \lambda_j | i \leq j\}$, so

$$\chi_{\mathrm{Sym}^2 V}(g) = \sum_{i \leq j} \lambda_i \lambda_j = \frac{1}{2}\left(\sum_i \lambda_i\right)^2 + \frac{1}{2}\sum_i \lambda_i^2 = \frac{1}{2}[\chi_V(g)^2 + \chi_V(g^2)].$$

   Similarly, the eigenvalues for the action of g on $\bigwedge V$ are $\{\lambda_i \lambda_j | i < j\}$, so

$$\chi_{\bigwedge^2 V}(g) = \sum_{i < j} \lambda_i \lambda_j = \frac{1}{2}\left(\sum_i \lambda_i\right)^2 - \frac{1}{2}\sum_i \lambda_i^2 = \frac{1}{2}[\chi_V(g)^2 - \chi_V(g^2)].$$

   By the above and Proposition 8, we have

$$\chi_{\mathrm{Sym}^2 V \oplus \bigwedge^2 V}(g) = \chi_{\mathrm{Sym}^2 V}(g) + \chi_{\bigwedge^2 V}(g) = \chi_V(g)^2 = \chi_{V \otimes V}(g).$$

7. Let $V := V_{\mathrm{ref}}$ be the reflection representation of S_3 Use the character table of S_3

	1	3	2
S_3	()	(12)	(123)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{ref}	2	0	-1

to determine the decomposition of $V^{\otimes n}, n \geq 1$, into irreducibles.

Solution. Let χ_n be the character of $V^{\otimes n}$ By Proposition 8, $\chi_n(g) = (\chi_{\text{ref}}(g))^n$, so $\chi_n(()) = 2^n, \chi_n((12)) = 0$, and $\chi_n((123)) = (-1)^n$. If $V = aV_{\text{triv}} \oplus aV_{\text{sgn}} \oplus cV_{\text{ref}}$, then matching characters gives

$$a + b + 2c = 2^n \qquad a - b = 0, \qquad a + b - c = (-1)^n,$$

and solving for a, b, and c, we obtain

$$a = b = \frac{2^{n-1} + (-1)^n}{3}, \qquad c = \frac{2^n - (-1)^n}{3}.$$

In particular, these are nonnegative integers since $2^n \equiv (-1)^n \mod 3$.