Group Representation Theory

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Adapted from the notes of a one-semester graduate course taught at the University of Illinois Urbana–Champaign.

About these notes

These notes are adapted from a one-semester graduate course, Math 506: Group Representation Theory, taught at the University of Illinois Urbana–Champaign in the spring of 2026. The course webpage containing the original handwritten notes can be found here. Claude code was used to TeX up those notes, as well as minor corrections and smoothing; it is possible that this process has introduced minor errors; it certainly caught a few.

Representation theory studies linear actions of a group, or of some other algebraic object, on a vector space. The central problem is to decompose arbitrary representations into irreducibles. All representations in this course are finite dimensional and over the complex numbers.

- **Part I** treats finite groups. Here there are only finitely many irreducibles, and by Maschke's theorem every representation is a direct sum of them; Schur's lemma computes the homomorphisms between irreducibles. The main tool is character theory. The irreducible characters form an orthonormal basis for the space of class functions, so the number of irreducibles is the number of conjugacy classes, and a representation is determined by its character. We then treat restriction, induction, and Frobenius reciprocity, and end with an application: the representation theory of $\mathrm{GL}_2(\mathbb{F}_q)$, sketched rather than proved in full.
- **Part II** specializes to the symmetric groups, where the representation theory can be expressed in terms of the combinatorics of partitions and Young tableaux. The irreducibles are the Specht modules, one for each partition of n. A Specht module sits inside a permutation module and is spanned by polytabloids; the standard polytabloids are a basis, so its dimension is the number of standard Young tableaux of the corresponding shape. The permutation modules themselves decompose with multiplicities the Kostka numbers. Restriction and induction between S_{n-1} and S_n are given by the branching rule, and the irreducible characters by the Murnaghan–Nakayama rule, which we state here and prove in Part III. The Specht modules are defined over any field; in positive characteristic they need not be irreducible, but their quotients by the radical of the natural bilinear form still give all the irreducibles.
- **Part III** turns to Lie algebras and Lie groups over $\mathbb{C}$. Since we do not assume any Lie theory

background, many proofs are sketched or omitted. The key example is the rank-1 Lie algebra $\mathfrak{sl}_2(\mathbb{C})$; a semisimple Lie algebra can be characterized by the ways in which copies of $\mathfrak{sl}_2(\mathbb{C})$ sit inside it. This gives the Killing–Cartan theorem, the classification of semisimple Lie algebras in terms of root systems, or, equivalently, Dynkin diagrams. The theorem of the highest weight parametrizes the irreducibles by dominant integral weights, and the Weyl character and dimension formulas make them explicit. On the group side, the matrix exponential relates a Lie group to its Lie algebra. Compact groups have complete reducibility, by Haar measure and the Peter–Weyl theorem, and the correspondence carries this back to semisimple Lie algebras and Lie groups. The same correspondence classifies the irreducibles of $\mathrm{GL}_n(\mathbb{C})$; the polynomial ones are indexed by partitions. The last three sections return to the symmetric groups. The double commutant theorem gives Schur–Weyl duality: the kth tensor power of the standard representation of $\mathrm{GL}_n(\mathbb{C})$ decomposes, under the commuting actions of $\mathrm{GL}_n(\mathbb{C})$ and S_k, into a sum over partitions of an irreducible for one group tensored with an irreducible for the other. The Frobenius characteristic map identifies the class functions on the symmetric groups with the symmetric functions, taking irreducible characters to Schur functions; the Murnaghan–Nakayama rule, promised in Part II, follows.

- **Part IV** consists of three independent topics. Invariant theory: Hilbert's finiteness theorem, the first and second fundamental theorems for the classical groups, Molien's formula, Chevalley–Shephard–Todd. Quiver representations: Gabriel's theorem, path algebras, bound quiver algebras. Quantum groups: Hopf algebras, $U_q(\mathfrak{sl}_2)$ and its representations, R-matrices and quasitriangularity, crystal bases.

The prerequisites are a graduate algebra course (groups, rings, modules, tensor products, and a little Galois theory for Section 8) and linear algebra. Point-set topology and measure theory are used lightly in Section 25, and no Lie theory is assumed.

Numbering

Definitions, theorems, lemmas, propositions and corollaries share one counter that runs from 1 to 104 through the whole document. Examples, remarks, exercises, problems, and displayed equations each have their own separate running counter.

Exercises and problems

Exercises were in-class activities during lecture. They appear in-line, and are mostly short examples and verifications.

Problems were the homework problems assigned in the course. They are collected at the end of each part. Most (but not all) of these problems were taken from or inspired by the cited sources.

Sources

No single source covers everything in these notes. The book that comes closest to being the default reference is Fulton–Harris [FH91]. Other key sources include Serre [Ser77] for Part I, Sagan [Sag01] and James [JK81] for Part II, Bump [Bum04], Humphreys [Hum72] and Hall [Hal15] for Part III, and several sources for Part IV. Individual sections cite what they follow. The bibliography lists these and other sources.

Contents

1. Introduction

A *representation* is a linear action of a group (or algebraic object) on a vector space; equivalently, it is a homomorphism

$$\rho\colon G \longrightarrow \mathrm{GL}(V) = \mathrm{GL}_n(F).$$

This is made precise in Definition 1.

A representation V is *irreducible* if whenever $W \leq V$ with $GW \subseteq W$, we have $W = \{0\}$ or $W = V$.

Main problem of representation theory: classify all irreps of G, and describe how arbitrary representations decompose into irreps.

Example 1. Let

$$S^1 = \{\, z \in \mathbb{C} \mid |z| = 1 \,\},$$

a group under multiplication, and let

$$L^2(S^1) = \left\{ f\colon S^1 \to \mathbb{C} \;\middle|\; \int_{S^1} |f(x)|^2 \, dx < \infty \right\}.$$

This is a representation via

$$(z \cdot f)(w) := f(wz).$$

S^1 is an abelian group, so its irreps are 1-dimensional: for $n \in \mathbb{Z}$,

$$\rho_n\colon S^1 \longrightarrow \mathbb{C}^\times = V_n, \qquad z \longmapsto z^n \in S^1.$$

The Peter–Weyl Theorem says that $L^2(G)$ decomposes as an orthogonal direct sum of irreps:

$$L^2(S^1) \;\cong\; \widehat{\bigoplus_{n\in\mathbb{Z}}} V_n,$$

where the inner product is $\langle f, g\rangle = \int_{S^1} f(x)\overline{g(x)}\, dx$.

Concretely, writing $f(x) := \underbrace{g}_{\in L^2(S^1)}\left(e^{2\pi i x}\right)$, we get

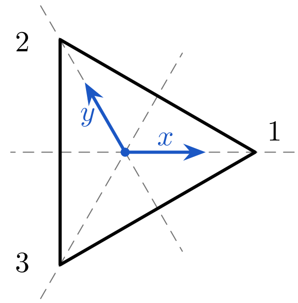
$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x} \qquad \text{Fourier series!}$$

where

$$c_n = \left\langle f, e^{2\pi i n\theta} \right\rangle = \int_0^1 f(x) e^{-2\pi i n x}\, dx.$$

Generalizing this picture leads to *harmonic analysis.*

Example 2. $G = S_3$, $F = \mathbb{C}$.



Irreps:

$$\rho_{\text{triv}} \colon w \longmapsto \begin{bmatrix}1\end{bmatrix},$$
$$\rho_{\text{sgn}} \colon w \longmapsto \begin{bmatrix}(-1)^w\end{bmatrix}.$$

And ρ_{ref}, in the basis x, y drawn above:

$$(1) \longmapsto \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad (12) \longmapsto \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$
$$(13) \longmapsto \begin{bmatrix}-1 & 0\\ -1 & 1\end{bmatrix} \qquad (23) \longmapsto \begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix}$$
$$(123) \longmapsto \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix} \qquad (132) \longmapsto \begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}$$

Consider the *regular representation*

$$V_{\text{reg}} = \operatorname{span}\{v_{(1)}, v_{(12)}, v_{(13)}, v_{(23)}, v_{(123)}, v_{(132)}\}$$

with the action

$$w \cdot v_u := v_{wu}.$$

It decomposes as a direct sum of irreps,

$$V_{\text{reg}} \;\cong\; V_{\text{triv}} \oplus V_{\text{sgn}} \oplus V_{\text{ref}} \oplus V_{\text{ref}},$$

an instance of the general statement in Corollary 13.

Example 3. $G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle$, $V = \mathbb{F}_p^2$, with

$$g^a \longmapsto \begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}, \qquad \text{since} \qquad \begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & a+b\\ 0 & 1\end{bmatrix}.$$

The subspace $W = \left\langle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \right\rangle$ is invariant, since

$$\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ 0\end{bmatrix} = \begin{bmatrix}x\\ 0\end{bmatrix} \in W,$$

but no other subspace is, since

$$\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x+ay\\ y\end{bmatrix} \notin \left\langle \left[\begin{smallmatrix}x\\y\end{smallmatrix}\right] \right\rangle \qquad \text{if } a \neq 0,\ y \neq 0.$$

So V *cannot* be written as a direct sum of irreps!

Example 4. A *quiver representation* is a directed graph of vector spaces and linear maps, for instance

$$\mathbb{C} \xrightarrow{\begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}} \mathbb{C}^3 \xrightarrow{\begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 2 \end{bmatrix}} \mathbb{C}^2.$$

Here we have two types of "atomic object".

Simples:

$$\mathbb{C} \longrightarrow 0 \longrightarrow 0 \qquad 0 \longrightarrow \mathbb{C} \longrightarrow 0 \qquad 0 \longrightarrow 0 \longrightarrow \mathbb{C}$$

Indecomposables: also include

$$\mathbb{C} \xrightarrow{[1]} \mathbb{C} \longrightarrow 0 \qquad 0 \longrightarrow \mathbb{C} \xrightarrow{[1]} \mathbb{C} \qquad \mathbb{C} \xrightarrow{[1]} \mathbb{C} \xrightarrow{[1]} \mathbb{C}$$

Quiver representations are closely related to the representation theory of finite-dimensional algebras.

PART I

Representations of finite groups

2. Representations

Definition 1.

a) A *representation* (ρ, V) of a group G is a homomorphism

$$\rho\colon G \longrightarrow \mathrm{GL}(V).$$

Equivalently, (ρ, V) is a linear (left) action of G on V, and V is a vector space that is also a (left) G-module. We will often use just ρ or V to express (ρ, V).

b) A *subrepresentation* of (ρ, V) is a subspace $W \leq V$ that is G-invariant: $GW \subseteq W$. Now $\{0\}$ and V are always subrepresentations; if there are no others, V is called *irreducible*.

c) If V and W are G-representations, a *G-equivariant map* or *G-module homomorphism* is a linear map $\varphi\colon V \to W$ such that the diagrams

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$

commute. If φ is a bijection it is an *isomorphism*.
Let $\mathrm{Hom}_G(V, W)$ be the vector space of G-equivariant maps $V \to W$, and $\mathrm{End}_G(V)$ be the ring $\mathrm{Hom}_G(V, V)$.

d) The *trivial representation* is $(\rho_{\mathrm{triv}}, V_{\mathrm{triv}})$ where $V_{\mathrm{triv}} = F$ (a field) and $\rho_{\mathrm{triv}}(g) = \begin{bmatrix}1\end{bmatrix}$ for all $g \in G$.

Suppose V and W are G-representations. The following are G-representations:

- *Direct sum:* $V \oplus W$, via $g \cdot (v + w) = g \cdot v + g \cdot w$.
- *Tensor product:* $V \otimes W$, via $g \cdot (v \otimes w) = gv \otimes gw$.
- *Tensor power:* $V^{\otimes n}$.
- *Symmetric power:* $\mathrm{Sym}^n V = V^{\otimes n} \big/ \big(v_i \otimes v_j - v_j \otimes v_i\big)$.
- *Exterior power:* $\bigwedge^n V = V^{\otimes n} \big/ \big(v_i \otimes v_j + v_j \otimes v_i\big)$.
- $\mathrm{Hom}(V, W)$, via
$$(g\varphi)(v) := g\varphi(g^{-1}v) \qquad \text{(trivial on } \mathrm{Hom}_G(V, W)\text{)}$$
- *Dual/contragredient:* $V^* = \mathrm{Hom}(V, V_{\mathrm{triv}})$, via
$$(g\varphi)(v) := \varphi(g^{-1}v) \quad \text{or} \quad \langle \rho^*(g)v^*, \rho(g)v\rangle = \langle v^*, v\rangle \quad \text{or} \quad \rho^*(g) = \rho(g^{-1})^{\mathsf{T}}.$$

Exercise 1. Let

$$\rho\colon\ \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{GL}_2(\mathbb{C}), \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the identity representation. Compute $\rho^{\otimes 2}$, $\mathrm{Sym}^2 \rho$, $\bigwedge^2 \rho$, and ρ^*.

Lemma 2. *Let $\varphi \in \mathrm{Hom}_G(V, W)$. Then* $\ker\varphi$, $\operatorname{im}\varphi$, *and* $\operatorname{coker}\varphi$ *are representations.*

Proof.

$$v \in \ker\varphi \implies \varphi(g\cdot v) = g\varphi(v) = g\cdot 0 = 0,$$
$$w \in \operatorname{im}\varphi \implies g\cdot w = g\underbrace{\varphi(v)}_{\exists v} = \varphi(gv) \in \operatorname{im}\varphi.$$

This also implies that

$$g\cdot \underbrace{(x+\operatorname{im}\varphi)}_{\in \operatorname{coker}\varphi} := gx + \operatorname{im}\varphi$$

is well-defined. □

In the case where $V \le W$ and φ is the inclusion map, $\operatorname{coker}\varphi = W/V$ is called the *quotient representation.*

3. Schur's lemma and Maschke's theorem

Definition 3. A representation V is *completely reducible* if it is (isomorphic to) a direct sum of irreps:

$$V = V_1 \oplus \cdots \oplus V_m = \overbrace{c_1}^{\text{multiplicities}} V_1 \oplus \cdots \oplus c_k V_k,$$

where c_1V_1 means $\underbrace{V_1 \oplus \cdots \oplus V_1}_{c_1}$ and the $V_1,\ldots,V_k$ are mutually non-isomorphic.

Recall that Example 3 was a representation that is not completely reducible.

Now for the next several weeks let us restrict to the case where G is a finite group and V is a finite-dimensional complex vector space.

Proposition 4 (Schur's Lemma)**.** *Let V, W be G-irreps and $\varphi \in \operatorname{Hom}_G(V,W)$.*

a) *Either φ is an isomorphism or $\varphi = 0$.*
b) *If $V = W$, then φ is a scalar multiple of the identity.*

Proof. a) $\ker\varphi$ and $\operatorname{im}\varphi$ are subrepresentations of V and W. Since V and W are irreps, we must have either

$$\underbrace{\ker\varphi = 0, \quad \operatorname{im}\varphi = W}_{\text{isom.}} \qquad \text{or} \qquad \underbrace{\ker\varphi = V, \quad \operatorname{im}\varphi = 0.}_{\varphi=0}$$

b) Since $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue λ. This means that $\ker(\varphi-\lambda I) \neq 0$, so by a), $\varphi - \lambda I$ is the zero map, and $\varphi = \lambda I$.

□

Theorem 5 (Maschke's Theorem)**.** *Every finite-dimensional complex representation V of a finite group is completely reducible. Moreover, the decomposition*

$$V = c_1V_1 \oplus \cdots \oplus c_kV_k$$

is unique up to isomorphism and the order of the summands.

Proof. If V is irreducible, we are done, so suppose V has a proper nontrivial subrepresentation W. We want to find a complementary G-invariant subspace, i.e. a subrepresentation U such that $V = W \oplus U$.

To do so, we use "Weyl's averaging trick" to construct a G-invariant inner product. Let $\langle \cdot, \cdot \rangle$ denote any Hermitian inner product on V (e.g. $\langle v, w \rangle = v^\top \overline{w}$). Define

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle .$$

Now $\langle \cdot, \cdot \rangle_G$ is also a Hermitian inner product:

- conjugate symmetric,
- linear in the first argument,
- positive definite.

We claim that $\langle \cdot, \cdot \rangle_G$ is G-invariant:

$$\langle gv, gw \rangle_G = \langle v, w \rangle_G \qquad \forall g \in G.$$

Indeed,

$$\langle gv, gw \rangle_G = \frac{1}{|G|} \sum_{h \in G} \langle hgv, hgw \rangle = \frac{1}{|G|} \sum_{h' \in G} \langle h'v, h'w \rangle = \langle v, w \rangle_G .$$

Let $U = W^\perp$, the orthogonal complement with respect to $\langle \cdot, \cdot \rangle_G$. Then $V = U \oplus W$ as vector spaces, so we only need to show that U is G-invariant. But if $u \in U$, then for all $w \in W$, $g \in G$,

$$\langle gu, w \rangle_G = \langle u, \underbrace{g^{-1}w}_{\in W} \rangle_G = 0,$$

so $gu \in U$, and U is a subrepresentation.

Now, since V is finite-dimensional, we use induction on dimension and write U, W as a direct sum of irreps, making V a direct sum of irreps too.

Finally, suppose

$$V \cong c_1 V_1 \oplus \cdots \oplus c_k V_k \cong d_1 W_1 \oplus \cdots \oplus d_\ell W_\ell$$

are two irreducible decompositions of V. Write the identity map in block matrix form

$$\begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1k} \\ \vdots & & \vdots \\ \varphi_{\ell 1} & \cdots & \varphi_{\ell k} \end{bmatrix} \qquad \text{where} \qquad \varphi_{ij} \colon c_i V_i \longrightarrow d_j W_j .$$

By Schur's Lemma (Proposition 4), $\varphi_{ij} = 0$ unless $V_i \cong W_j$, and then we must also have $c_i = d_j$ and φ_{ij} is an isomorphism $c_i V_i \to d_j W_j$. Thus $k = \ell$ also, and the two decompositions are equivalent. □

Remark 1. This argument works over most fields, but fails over $\mathbb{F}_p$ when $p \mid G$. Recall the example

$$G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle, \qquad V = \mathbb{F}_p^2, \qquad g^a \longmapsto \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \qquad W = \left\langle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \right\rangle \colon \text{invariant subspace.}$$

Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{F}_p^2$ and let

$$\langle v, w \rangle_G = \sum_{g \in G} \langle gv, gw \rangle .$$

Then

$$\left\langle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \right\rangle_G = \sum_{a \in \mathbb{F}_p} \left\langle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \right\rangle = p \cdot 1 = 0,$$

which would imply that $W \subseteq W^\perp$, i.e. $\langle \cdot, \cdot \rangle_G$ is not an inner product.

Remark 2. Note that the uniqueness only holds up to isomorphism. For example, with $V = V_{\text{triv}} \oplus V_{\text{triv}}$,

$$V = \left\langle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \right\rangle \oplus \left\langle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \right\rangle = \left\langle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \right\rangle \oplus \left\langle \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \right\rangle = \left\langle \left[\begin{smallmatrix}2\\3\end{smallmatrix}\right] \right\rangle \oplus \left\langle \left[\begin{smallmatrix}-2\\5\end{smallmatrix}\right] \right\rangle$$

are all valid irreducible decompositions.

4. Characters

Proposition 6. *If G is abelian, then every G-irrep is one-dimensional.*

Proof. First we claim that $\rho(g) \in \mathrm{End}_G(V)$ for all $g \in G$. Indeed, if $h \in G$, $v \in V$, then since G is abelian,

$$\rho(g)\rho(h)v = \rho(h)\rho(g)v,$$

so $\rho(g)$ is G-equivariant.

By Schur's Lemma, every $\rho(g)$ is a scalar multiple of the identity. This means that every subspace of V is G-invariant, so V must be 1-dimensional. $\square$

Example 5. $G = \mathbb{Z}/n\mathbb{Z} = \langle g \rangle$. Irreps of G are of the form

$$\rho_\zeta : G \longrightarrow \mathbb{C}^\times, \qquad g^a = \zeta^a, \qquad \zeta: n\text{th root of } 1.$$

So there are exactly n non-isomorphic irreps of G!

$G = \mathbb{Z}/4\mathbb{Z}$	$\rho(1)$	$\rho(g)$	$\rho(g^2)$	$\rho(g^3)$
ρ_1	1	1	1	1
ρ_i	1	i	-1	$-i$
ρ_{-1}	1	-1	1	-1
ρ_{-i}	1	$-i$	-1	i

Very nice looking table! (square, roots of unity, orthogonal rows / cols.) And it tells us everything we could want to know about the representation theory of G.

Goal: get as close as we can to this for all finite groups.

Definition 7. The *character* of a representation (ρ, V) is the function $\chi_V : G \to \mathbb{C}$ given by the trace of ρ:

$$\chi_V(g) = \mathrm{tr}\big(\rho(g)\big).$$

Proposition 8.

a) *The character is a* class function*:* $\chi_V(hgh^{-1}) = \chi_V(g)$.

b) $\chi_V(1) = \dim V$.

c) $\chi_{V\oplus W} = \chi_V + \chi_W$.

d) $\chi_{V\otimes W} = \chi_V\chi_W$.

e) $\chi_{V^*} = \overline{\chi_V}$.

f) $\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}\big[\chi_V(g)^2 + \chi_V(g^2)\big]$.

g) $\chi_{\bigwedge^2 V}(g) = \frac{1}{2}\big[\chi_V(g)^2 - \chi_V(g^2)\big]$.

Proof. a) Trace is a class function.

b) $\rho(1) = \mathrm{id}_V$.

c), d), e) Let $g \in G$, and let $\lambda_1, \dots, \lambda_k$ be the eigenvalues of $\rho_V(g)$, and $\mu_1, \dots, \mu_\ell$ be the eigenvalues of $\rho_W(g)$. Then:

c) The eigenvalues of $\rho_{V\oplus W}(g) = \rho_V(g) \oplus \rho_W(g)$ are $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_\ell$.

d) The eigenvalues of $\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$ are $\lambda_i\mu_j$, $1 \le i \le k$, $1 \le j \le \ell$.

e) The eigenvalues of $\rho_{V^*}(g) = \rho_V(g^{-1})^\intercal$ are $\lambda_1^{-1}, \dots, \lambda_k^{-1}$. Since $\rho_V(g)$ has finite order, its eigenvalues are roots of unity, so $\lambda_i^{-1} = \overline{\lambda_i}$, and so

$$\chi_{V^*}(g) = \overline{\lambda_1} + \cdots + \overline{\lambda_k} = \overline{\chi_V(g)}.$$

f), g) are Problem 6. □

We collect the characters of irreps of G into the *character table*:

- Rows indexed by irreps.
- Columns indexed by conjugacy classes in G.
- Entries are the character value for the irrep at (any element of) the conjugacy class.

Example 6. Recall from Section 1 the three irreps of S_3; by Corollary 14 below there are no others.

$$\rho_{\mathrm{triv}} \colon w \longmapsto \begin{bmatrix}1\end{bmatrix}, \qquad \rho_{\mathrm{sgn}} \colon w \longmapsto \begin{bmatrix}(-1)^w\end{bmatrix},$$

ρ_{ref} :

$$(1) \longmapsto \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad (23) \longmapsto \begin{bmatrix}1 & -1\\ 0 & -1\end{bmatrix}$$

$$(12) \longmapsto \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \qquad (123) \longmapsto \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}$$

$$(13) \longmapsto \begin{bmatrix}-1 & 0\\ -1 & 1\end{bmatrix} \qquad (132) \longmapsto \begin{bmatrix}-1 & 1\\ -1 & 0\end{bmatrix}$$

Conjugacy classes are determined by cycle type, so the character table is:

S_3	1 elt $()$	3 elts (12)	2 elts (123)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{ref}	2	0	-1

Exercise 2. Compute the characters of the following representations of S_3. Can they be written as sums of irreducible characters?

a) $\rho_{\text{triv}} \oplus \rho_{\text{sgn}}$
b) ρ^*_{ref}
c) $\rho^{\otimes 2}_{\text{ref}}$
d) $\operatorname{Sym}^2 \rho_{\text{ref}}$
e) $\bigwedge^2 \rho_{\text{ref}}$
f) The *regular representation* $V_{\text{reg}} = \langle v_g \mid g \in S_3 \rangle$ with the action $w \cdot v_u := v_{wu}$, $w \in S_3$.

5. Orthogonality of characters

For any G-representation V, let V^G be the isotypic component of the trivial representation.

Proposition 9. *The map*

$$\varphi := \frac{1}{|G|} \sum_{g \in G} g \ \in \ \operatorname{End}_G V$$

is a projection $V \to V^G$. Furthermore, the multiplicity of the trivial representation inside V is given by

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g). \tag{1}$$

Proof. If $v \in V$, then for all $h \in G$,

$$h\varphi(v) = \frac{1}{|G|} \sum_{g \in G} hgv = \frac{1}{|G|} \sum_{g' \in G} g'v = \varphi(v),$$

so $\operatorname{im} \varphi \subseteq V^G$. On the other hand, if $v \in V^G$, then

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} v = v,$$

so $V^G \subseteq \operatorname{im} \varphi$ and $\varphi^2 = \varphi$.

For the multiplicity of the trivial representation, since φ is a projection,

$$\dim V^G = \operatorname{tr}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g). \qquad \square$$

We will use this to prove orthogonality of characters.

Let $\mathbb{C}_{\mathrm{cl}}(G)$ be the space of class functions $G \to \mathbb{C}$. We have a Hermitian inner product on $\mathbb{C}_{\mathrm{cl}}(G)$:

$$(\alpha, \beta) := \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\, \beta(g) \qquad \text{(convention note: linear in the \textit{second} spot)}$$

Theorem 10. *The characters of G-irreps form an orthonormal set in $\mathbb{C}_{\mathrm{cl}}(G)$.*

Proof. Let V, W be G-irreps. We apply the previous result to $\mathrm{Hom}(V, W)$. By Problem 1a) and Proposition 8,

$$\chi_{\mathrm{Hom}(V,W)} = \chi_{V^* \otimes W} = \overline{\chi_V}\, \chi_W.$$

On the other hand, by Problem 1b), $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$. By Schur's Lemma, we have

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases}$$

Applying (1) now gives

$$(\chi_V, \chi_W) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases}$$ □

Corollary 11. *The number of irreps of G is $\leq$ the number of conjugacy classes.*

Corollary 12. *If V, W have irreducible decompositions*

$$V \cong c_1 V_1 \oplus \cdots \oplus c_k V_k, \qquad W \cong d_1 V_1 \oplus \cdots \oplus d_k V_k,$$

then

$$(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(V, W) = \sum_i c_i d_i.$$

In particular:

- *if V is irreducible, its multiplicity in W is (χ_V, χ_W);*
- *V is irreducible $\iff$ $(\chi_V, \chi_V) = 1$;*
- *every representation is determined by its character.*

Let V_{reg} be the regular representation of G (see Problem 4).

Corollary 13. *Every G-irrep V appears in V_{reg} with multiplicity $\dim V$. That is, if $V_1, \ldots, V_k$ are the G-irreps,*

$$V_{\mathrm{reg}} \cong \bigoplus_i (\dim V_i) V_i. \tag{2}$$

Proof. Since the action of a non-identity element has no fixed points,

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If V is an irrep, the multiplicity of V in V_{reg} is

$$(\chi_V, \chi_{\mathrm{reg}}) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \chi_{\mathrm{reg}}(g) = \dim V. \qquad \square$$

Taking the character of both sides of (2), we get:

Corollary 14.
$$|G| = \sum_i (\dim V_i)^2,$$
and for $g \neq e$,
$$\sum_i (\dim V_i)\, \chi_{V_i}(g) = 0. \qquad \text{(special case of \textit{column orthogonality})}$$

6. Character tables, restriction and induction

Proposition 15. *The number of irreps of G equals the number of conjugacy classes of G.*

Proof. By Corollary 11, we already know $\leq$. For $\geq$, suppose $\alpha \in \mathbb{C}_{\mathrm{cl}}(G)$ and $(\alpha, \chi_V) = 0$ for all irreps V. We will show that $\alpha = 0$, and the result follows.

For any G-representation V, set

$$\varphi_V := \sum_{g \in G} \alpha(g)\, \rho_V(g) \;\in\; \operatorname{End} V.$$

One can check that in fact $\varphi_V \in \operatorname{End}_G V$ [FH91, Prop. 2.28]

By Schur's Lemma, $\varphi_V = \lambda \cdot \mathrm{id}$, so

$$\lambda = \frac{1}{\dim V} \operatorname{tr}(\varphi_V) = \frac{1}{\dim V} \sum_{g \in G} \alpha(g)\, \chi_V(g) = \frac{|G|}{\dim V} (\alpha, \chi_{V^*}) = 0.$$

So $0 = \varphi_V = \sum_{g \in G} \alpha(g)\, \rho_V(g)$ for any G-representation V. But if $V = V_{\mathrm{reg}}$, then $\{\rho_V(g), g \in G\}$ are linearly independent matrices (proof: $\rho_V(g) v_e = v_g$, and the v_g's are linearly independent vectors). Therefore $\alpha = 0$. $\square$

Corollary 16. *The irreducible characters form an orthonormal basis of $\mathbb{C}_{\mathrm{cl}}(G)$. Furthermore, the columns of the character table are also orthogonal:*

$$\sum_{\chi \text{ irred}} \overline{\chi(g)}\, \chi(h) = \begin{cases} |\text{centralizer of } g|, & \text{if } g = h, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The first statement follows from Theorem 10 and Proposition 15. Column orthogonality is a consequence of squareness and row orthogonality. The details are left to the reader; see Problem 9. $\square$

Example 7. S_4.

S_4	1 $()$	6 (12)	8 (123)	6 (1234)	3 $(12)(34)$
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_{ref}	3	1	0	-1	-1
$\chi_{\text{ref}} \otimes \chi_{\text{sgn}}$	3	-1	0	1	-1
χ_W	2	0	-1	0	2

Notes: $V_{\text{perm}} = \mathbb{C}^4$, $w \cdot v_i := v_{w(i)}$, has character

$$\chi_{\text{perm}} : \; \begin{array}{|ccccc} 4 & 2 & 1 & 0 & 0 \end{array}$$

which decomposes as $\chi_{\text{triv}} + (\text{something})$, and use the inner product to show that the something is irreducible. For W: use column orthogonality.

What about A_4?

A_4	1 $()$	4 (123)	4 (132)	3 $(12)(34)$
χ_{triv}	1	1	1	1
χ_{ref}	3	0	0	-1
χ	1	ζ	ζ^2	1
χ'	1	ζ^2	ζ	1

Notes:

$$\chi_W : \; \begin{array}{|cccc} 2 & -1 & -1 & 2 \end{array}$$

has $(\chi_W, \chi_W) = 2$; use orthogonality to compute the characters.

We obtained characters of A_4 by "restricting" characters of S_4. We now go the other way.

Definition 17. Let $H \leq G$.

a) If (ρ, V) is a G-representation, the *restriction* of (ρ, V) to H is the H-representation

$$\operatorname{Res}^G_H(\rho, V) = (\rho|_H, V) \qquad (\text{or } \operatorname{Res}^G_H V, \text{ or } \operatorname{Res}^G_H \rho)$$

b) Let $\sigma_1, \dots, \sigma_k$ be a set of representatives for G/H. If (π, W) is an H-representation, the *induced representation* of (ρ, W) to G is the G-representation

$$\operatorname{Ind}^G_H(\pi, W) = (\rho, V)$$

where

$$V := \bigoplus_{i=1}^k W_i$$

and

$$\rho(g) \underset{\in W_i}{w_i} = \big(\pi(h_i)w\big)_j, \qquad \text{where } g \underset{\in G/H}{\sigma_i} = \sigma_j \underset{\in H}{h}.$$

Also note that

$$\dim \operatorname{Ind}_H^G W = |G : H| \dim W$$

and, by *Mackey's formula* [Ser77],

$$\operatorname{Res}_H^G \operatorname{Ind}_H^G W \;\cong \bigoplus_{HgH \,\in\, H\backslash G/H} \operatorname{Ind}_{H\cap {}^gH}^H \big({}^gW\big),$$

where ${}^gH = gHg^{-1}$ and ${}^gW$ is W made into a representation of $H \cap {}^gH$ via $x \mapsto g^{-1}xg$.

Example 8.

a) $V_{\mathrm{reg}} = \operatorname{Ind}_1^G V_{\mathrm{triv}}$.

b) More generally,

$$\operatorname{Ind}_H^G V_{\mathrm{triv}}$$

is the *permutation representation* (Problem 3) of the action of G on G/H:

$$\underset{\in G}{\sigma h} \cdot v_\tau := v_{\sigma\tau}, \qquad \tau \in G/H,\ h \in H.$$

7. Frobenius reciprocity

Recall Definition 17b): for $H \le G$ with coset representatives $\sigma_1, \ldots, \sigma_k$ for G/H and (π, W) an H-representation, $\operatorname{Ind}_H^G(\pi, W) = (\rho, V)$ with $V = \bigoplus_{i=1}^k W_i$ and $\rho(g)w_i = (\pi(h)w)_j$ where $g\sigma_i = \sigma_j h$.

Ind_H^G does not depend on the choice of coset representatives (up to isomorphism).

Proof. See Problem 11; alternatively, it follows from Frobenius reciprocity (Theorem 19), which characterizes $\operatorname{Ind}_H^G W$ up to isomorphism by a universal property. □

Also note that

- $\operatorname{Ind}_H^G(W_1 \oplus W_2) = \operatorname{Ind}_H^G W_1 \oplus \operatorname{Ind}_H^G W_2$;
- $\operatorname{Ind}_K^G \operatorname{Ind}_H^K W = \operatorname{Ind}_H^G W$.

Example 9. Continuing the previous example, let $H = \langle (12) \rangle \le S_3$, with coset representatives

$$\sigma_1 = (), \qquad \sigma_2 = (13), \qquad \sigma_3 = (23).$$

Then

$$V := \operatorname{Ind}_H^G V_{\mathrm{triv}} = \big\langle v_{()},\, v_{(13)},\, v_{(23)} \big\rangle,$$

with

$$() \longmapsto \mathrm{id}_V$$

and

$$\begin{aligned}
(12)\, v_{()} &= v_{()}, \\
(12)\, v_{(13)} &= (12)(13)\, v_{()} = (123)\, v_{()} = (23)(12)\, v_{()} = (23)\, v_{()} = v_{(23)}, \\
(12)\, v_{(23)} &= (12)(23)\, v_{()} = (132)\, v_{()} = (13)(12)\, v_{()} = (13)\, v_{()} = v_{(13)},
\end{aligned}$$

$$\begin{aligned}(13)\, v_{()} &= v_{(13)},\\ (13)\, v_{(13)} &= v_{()},\\ (13)\, v_{(23)} &= (13)(23)\, v_{()} = (132)\, v_{()} = (23)(12)\, v_{()} = (23)\, v_{()} = v_{(23)},\end{aligned}$$

$$(123)\, v_{(13)} = (123)(13)\, v_{()} = (23)\, v_{()} = v_{(23)},$$

and so on.

Now, fix G/H coset representatives $\sigma_1, \dots, \sigma_k$, taking $\sigma_1 = 1$. As an H-representation, $hw_1 = (hw)_1$, so we can identify W with $W_1 \leq \operatorname{Ind}_H^G W$.

Proposition 18. *Let $V = \operatorname{Ind}_H^G W$. Then*

$$\chi_V(g) = \frac{1}{|H|} \sum_{\substack{a \in G \\ a^{-1}ga \in H}} \chi_W(a^{-1}ga).$$

Proof. Since $V = \bigoplus_{i=1}^k W_i$, we have $\chi_V(g) = \sum_i \operatorname{tr} g|_{W_i}$.

Let $g\sigma_i = \sigma_j h$, $h \in H$; if $i \neq j$, then $\operatorname{tr}_{W_i} g = 0$. The cases where $g\sigma_i = \sigma_i h$ are exactly those i where $\sigma_i^{-1} g \sigma_i \in H$, and in that case $gW_i \subseteq W_i$ and $gw_i = (\sigma_i^{-1} g \sigma_i w)_i$, so

$$\operatorname{tr} g|_{W_i} = \operatorname{tr} \sigma_i^{-1} g \sigma_i|_W.$$

Thus

$$\chi_V(g) = \sum_{\sigma_i^{-1} g \sigma_i \in H} \chi_W(\sigma_i^{-1} g \sigma_i).$$

The formula follows from the fact that if $\sigma_i^{-1} g \sigma_i \in H$ and $\sigma_i H = aH$, then $a^{-1}ga \in H$ and $\chi_W(a^{-1}ga) = \chi_W(\sigma_i^{-1} g \sigma_i)$. ☐

Theorem 19 (Frobenius reciprocity)**.** *Let W be an H-representation and V be a G-representation. There is a natural vector space equivalence*

$$\operatorname{Hom}_H\big(W, \operatorname{Res}_H^G V\big) \;\cong\; \operatorname{Hom}_G\big(\operatorname{Ind}_H^G W, V\big).$$

Equivalently,

$$\big(\chi_W, \chi_{\operatorname{Res}_H^G V}\big)_H \;\cong\; \big(\chi_{\operatorname{Ind}_H^G W}, \chi_V\big)_G.$$

If W, V are irreducible, the multiplicity of W in $\operatorname{Res}_H^G V$ equals the multiplicity of V in $\operatorname{Ind}_H^G W$.

Proof. The equivalence of the two statements, and the last statement as a consequence of the second, are by Corollary 12 (or just by Schur's Lemma).

For the first statement:

- If $\phi \in \operatorname{Hom}_G\big(\operatorname{Ind}_H^G W, V\big)$, then $\phi|_{W_1} \in \operatorname{Hom}_H\big(W, \operatorname{Res}_H^G V\big)$.
- If $\varphi \in \operatorname{Hom}_H\big(W, \operatorname{Res}_H^G V\big)$, then $\phi \in \operatorname{Hom}_G\big(\operatorname{Ind}_H^G W, V\big)$, where

$$\phi|_{W_i} := \sigma_i \varphi \sigma_i^{-1}, \qquad w_i \longmapsto w \longmapsto \varphi(w) \longmapsto \sigma_i\big[\varphi(w)\big].$$

This is independent of the choice of representatives, since $\sigma_i h \varphi h^{-1} \sigma_i^{-1} = \sigma_i \varphi \sigma_i^{-1}$ (using $h\varphi = \varphi h$).

This is G-equivariant since if $g \in G$ satisfies $g\sigma_i = \sigma_j h$, $h \in H$, then

$$\begin{aligned} g\phi(w_i) &= g\sigma_i\big[\varphi(w)\big] = \sigma_j h\big[\varphi(w)\big] = \sigma_j\left[\varphi(hw)\right] && (H\text{-equivariance}) \\ &= \phi\big([hw]_j\big) = \phi(g \cdot w_i). \end{aligned}$$

□

8. The representations of $\mathrm{GL}_2(\mathbb{F}_q)$

Let $k = \mathbb{F}_q$ and $G = \mathrm{GL}_2(k)$. Set

$$B = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \,\middle|\, a, d \in k^\times,\ b \in k \right\} \subseteq G \qquad \text{Borel subgroup,}$$

$$T = \left\{ \begin{pmatrix} a & \\ & d \end{pmatrix} \,\middle|\, a, d \in k^\times \right\} \subseteq B \qquad \text{maximal (split) torus,}$$

$$U = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \,\middle|\, b \in k \right\} \subseteq B \qquad \text{unipotent subgroup.}$$

We have $B = U \rtimes T$ and $|B| = (q-1)^2 q$.

G acts transitively on the $q+1$ one-dimensional subspaces of k^2 ($\mathbb{P}^1(k)$), and B is the stabilizer of $\left\langle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \right\rangle \subseteq k^2$. So $|G/B| = q+1$, which implies

$$|G| = (q-1)^2 q(q+1).$$

Let $L = \mathbb{F}_q(\sqrt{D})$ be the unique quadratic extension of k. Then

$$L^\times \cong \left\{ \begin{pmatrix} x & Dy \\ y & x \end{pmatrix} \,\middle|\, x, y \text{ not both } 0 \right\} \subseteq G.$$

If $\varsigma = x + y\sqrt{D} \in L$, then

$$\operatorname{tr}\begin{pmatrix} x & Dy \\ y & x \end{pmatrix} = 2x = \operatorname{tr}_{L/k}\varsigma, \qquad \det\begin{pmatrix} x & Dy \\ y & x \end{pmatrix} = x^2 - Dy^2 = N_{L/k}\varsigma.$$

Conjugacy classes: Jordan form. Four cases:

a) 1 eigenvalue $x \in k$, diagonalizable;
b) 1 eigenvalue $x \in k$, not diagonalizable;
c) distinct eigenvalues $x, y \in k$ ($x \neq y$);
d) distinct eigenvalues $x \pm y\sqrt{D} \in L \setminus k$.

Representative	Num. elts. in class	Num. classes
$a_x = \begin{pmatrix} x & \\ & x \end{pmatrix}$	1	$q-1$
$b_x = \begin{pmatrix} x & 1 \\ & x \end{pmatrix}$	q^2-1	$q-1$
$c_{x,y} = \begin{pmatrix} x & \\ & y \end{pmatrix}$	q^2+q	$\frac{1}{2}(q-1)(q-2)$
$d_{x,y} = \begin{pmatrix} x & Dy \\ y & x \end{pmatrix}$	q^2-q	$\frac{1}{2}(q^2-q)$
Total:	$(q-1)^2 q(q+1)$ elts.	q^2-1 conj. classes

Principal series representations. *Parabolic induction:*

- T is abelian, so all irreps are 1-dimensional:

$$\mu_{\alpha,\beta}\begin{pmatrix} a & \\ & c \end{pmatrix} := \alpha(a)\beta(c), \qquad \alpha, \beta \colon k^\times \to \mathbb{C}^\times.$$

- "Inflate" to B: $B \twoheadrightarrow B/U \cong T$,

$$\rho'_{\alpha,\beta}\big(\underset{\in U}{u}\ \underset{\in T}{t}\big) := \mu_{\alpha,\beta}(t).$$

- Induce to G:

$$\rho_{\alpha,\beta} := \mathrm{Ind}_B^G \rho'_{\alpha,\beta}. \qquad \text{(might be reducible)}$$

Applying Proposition 18, its character is

	a_x	b_x	$c_{x,y}$	$d_{x,y}$
$\chi_{\alpha,\beta}\colon$	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0

So $\rho_{\alpha,\beta} \cong \rho_{\beta,\alpha}$, and otherwise they are non-isomorphic. One can compute

$$(\chi_{\alpha,\beta}, \chi_{\alpha,\beta}) = \begin{cases} 1, & \text{if } \alpha \neq \beta, \\ 2, & \text{if } \alpha = \beta, \end{cases}$$

so $\rho_{\alpha,\beta}$ is irreducible if $\alpha \neq \beta$.

Let

$$\rho_{\det,\alpha}\colon G \longrightarrow \mathbb{C}, \qquad g \longmapsto \alpha(\det g) \qquad \text{($q-1$ such representations)}$$

This has character (and, since it is 1-dimensional, equals)

	a_x	b_x	$c_{x,y}$	$d_{x,y}$
$\chi_{\det,\alpha}\colon$	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(x^2-Dy^2)$

And we can compute $(\chi_{\alpha,\alpha}, \chi_{\det,\alpha}) = 1$. So

$$\rho_{\alpha,\alpha} = \rho_{\det,\alpha} \oplus \widetilde{\rho}_\alpha \qquad \text{(q-dimensional)}$$

where $\widetilde{\rho}_\alpha$ is the irrep with character

	a_x	b_x	$c_{x,y}$	$d_{x,y}$
$\widetilde{\chi}_\alpha$:	$q\,\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(x^2-Dy^2)$

In the "principal series" (induced from 1-dimensional representations of B), we have

- $q-1$ irreps $\rho_{\det,\alpha}$ of dimension 1;
- $q-1$ irreps $\widetilde{\rho}_\alpha$ of dimension q;
- $\frac{1}{2}(q-1)(q-2)$ irreps $\rho_{\alpha,\beta}$ ($\alpha\neq\beta$) of dimension $q+1$.

Remaining: $\frac{1}{2}(q^2-q)$ irreps.

Cuspidal representations. We have obtained all the irreps we can arising from irreps of

$$T=\left\{\begin{pmatrix}x & \\ & y\end{pmatrix}\,\middle|\, x,y\in k^\times\right\}.$$

As a k-vector space,

$$\left\{\begin{pmatrix}x & Dy\\ y & x\end{pmatrix}\,\middle|\, x,y\in k^\times\right\}\;\cong\; T,$$

and this sits inside $L^\times$ (non-split torus).

So let us look at 1-dimensional representations of $L^\times$; there are $|L^\times|=q^2-1$ of them. If ρ is a representation of $k^\times$, we obtain a representation of $L^\times$:

$$\widehat{\rho}(\underset{\in L}{\ell}):=\rho\big(N_{L/k}(\ell)\big)\qquad (q-1\text{ representations of this form})$$

Call a 1-dimensional representation of $L^\times$ *indecomposable* if it does not factor through $N_{L/k}$ in this way. These are exactly the 1-dimensional representations ν of $L^\times$ such that

$$\overline{\nu}(\ell):=\nu(\overline{\ell})\qquad\text{satisfies}\qquad\overline{\nu}\neq\nu.$$

We would like to induce these representations up to G, but it is not (nearly) that simple. Let ν be an indecomposable 1-dimensional representation of $L^\times$ and put $\alpha=\nu|_{k^\times}$. Then

$$\widetilde{\rho}_1\otimes\rho_{\alpha,1}\;\cong\;\rho_{\alpha,1}\oplus\operatorname{Ind}_{L^\times}^G\nu\oplus\rho_\nu,\tag{3}$$

where ρ_ν is an irrep with character

	a_x	b_x	$c_{x,y}$	$d_{x,y}$
χ_ν	$(q-1)\nu(x)$	$-\nu(x)$	0	$-\nu(x+y\sqrt{D})-\overline{\nu}(x+y\sqrt{D})$

Equivalently, (3) says that ρ_ν is the virtual representation $\widetilde{\rho}_1\otimes\rho_{\alpha,1}-\rho_{\alpha,1}-\operatorname{Ind}_{L^\times}^G\nu$; that this is an honest irreducible representation is Problem 13.

Let

$$\underset{\|}{W:=L^\times\rtimes C_2},\qquad\langle\sigma\rangle,\qquad\sigma\ell=\overline{\ell}\,\sigma\qquad(\text{Weil group})$$

Consider the 2-dimensional representations τ of W. Since $L^\times$ is abelian,

$$\tau|_{L^\times} = \nu_1 \oplus \nu_2.$$

Case 1: τ is irreducible. τ_ν
One shows (Problem 14a) that $\overline{\nu_1} = \nu_2$ and that $\nu := \nu_1$ is indecomposable. $\frac{1}{2}q(q-1)$ of these

Case 2: τ is reducible. τ_{μ_1,μ_2}
One shows (Problem 14b) that ν_1 and ν_2 factor through $N_{L/k}$:

$$\nu_i(\ell) = \underbrace{\mu_i}_{\text{some 1D } k^\times \text{ repn.}}\big(N_{L/k}(\ell)\big).$$

$\frac{1}{2}q(q-1)$ of these

So we have a correspondence between all 2-dimensional representations of W and the higher-dimensional irreps of G:

$$\tau_\nu \longleftrightarrow \rho_\nu, \qquad\qquad \tau_{\alpha,\beta} \longleftrightarrow \begin{cases} \rho_{\alpha,\beta}, & \text{if } \alpha \neq \beta, \\ \widetilde{\rho}_\alpha, & \text{if } \alpha = \beta. \end{cases}$$

Problems

Unless otherwise stated, all groups here are finite and all representations are finite-dimensional and complex. All assertions require proof unless the problem says otherwise.

Problem 1. Let V and W be G-representations.

a) In Section 2 we defined G-actions on duals and on tensor products; combining these gives $V^* \otimes W$ the structure of a G-representation. Under the vector space identification

$$V^* \otimes W \cong \mathrm{Hom}(V, W), \qquad\qquad (\varphi \otimes w)(v) := \varphi(v)\, w,$$

we obtain a G-action on $\mathrm{Hom}(V, W)$. Show that this is the same representation as the one defined on $\mathrm{Hom}(V, W)$ in Section 2.

b) Show that $\phi \in \mathrm{Hom}(V, W)$ lies in the isotypic component of the trivial representation if and only if it is G-equivariant.

Problem 2. The *permutation representation* V_{perm} of S_3 is its permutation action on $\mathbb{C}^3$:

$$w \cdot e_i := e_{w(i)}, \qquad i = 1, 2, 3.$$

Without using character theory, show that $V_{\mathrm{perm}} \cong V_{\mathrm{triv}} \oplus V_{\mathrm{ref}}$. That is, find a one-dimensional fixed subspace (i.e. a copy of the trivial representation), and a complementary two-dimensional subspace that is S_3-invariant and irreducible — and therefore is the reflection representation, which you may assume is the only two-dimensional irreducible representation of S_3.

Problem 3. This generalizes the setting of the previous problem. Let X be a finite set with a G-action. The *permutation representation* associated to this action is the vector space $V = V_X = \{ v_x \mid x \in X \}$, with the action $gv_x := v_{g\cdot x}$.

a) Show that $\chi_V(g)$ is the number of elements of X fixed by g.
b) Show that the number of distinct G-orbits in X equals the multiplicity of the trivial representation inside V.

Problem 4. Let G be a finite group. Prove that the following representations are isomorphic:

$$\begin{aligned} R_1 &= \langle v_g \mid g \in G \rangle, & g \cdot v_h &:= v_{gh}; \\ R_2 &= \langle v_g \mid g \in G \rangle, & g \cdot v_h &:= v_{hg^{-1}}; \\ R_3 &= \langle f\colon G \to \mathbb{C} \rangle, & g \cdot f(h) &:= f(g^{-1}h); \\ R_4 &= \langle f\colon G \to \mathbb{C} \rangle, & g \cdot f(h) &:= f(hg). \end{aligned}$$

Any of these is called the *regular representation* of G. Here R_1 is the permutation representation associated to the left action of G on itself, while R_2 is the one associated to the right action. R_1 and R_3 are each called the *left regular representation*, while R_2 and R_4 are called the *right regular representation*. The definitions in terms of functions tend to be more useful when working with infinite groups.

Problem 5. Let V be an irreducible representation of the finite group G. In the proof of Maschke's theorem (Theorem 5) we constructed a G-invariant Hermitian inner product on V. Show that this inner product is unique up to scalar multiple. (Hint: Hermitian inner products on $\mathbb{C}^n$ are classified by $n \times n$ positive-definite Hermitian matrices.)

Problem 6. Show that for any representation V,

$$\chi_{\mathrm{Sym}^2 V}(g) = \tfrac{1}{2}\big[\chi_V(g)^2 + \chi_V(g^2)\big] \qquad \text{and} \qquad \chi_{\bigwedge^2 V}(g) = \tfrac{1}{2}\big[\chi_V(g)^2 - \chi_V(g^2)\big].$$

Use this to show that $V \otimes V \cong \mathrm{Sym}^2 V \oplus \bigwedge^2 V$.

Problem 7. Let $V := V_{\mathrm{ref}}$ be the reflection representation of S_3. Use the character table of S_3,

S_3	1 elt ()	3 elts (12)	2 elts (123)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{ref}	2	0	-1

to determine the decomposition of $V^{\otimes n}$, $n \geq 1$, into irreducibles.

Problem 8. Let V and W be G-representations, with W irreducible. Prove that the following map $V \to V$ is a G-equivariant projection onto the W-isotypic component of V:

$$\pi_W := \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \cdot g.$$

(In the case where W is the trivial representation this is Proposition 9.)

Problem 9. Prove the last statement of Corollary 16. That is, given that the irreducible characters form an orthonormal basis of $\mathbb{C}_{\mathrm{cl}}(G)$ with respect to the Hermitian inner product $(\cdot,\cdot)$, show that

$$\sum_{\chi\colon \text{irred}} \overline{\chi(g)}\,\chi(h) = \begin{cases} |C(g)|\,, & \text{if } g \sim h, \\ 0, & \text{otherwise,} \end{cases}$$

where $C(g)$ is the centralizer of g in G. (Hint: use properties of unitary matrices.)

Problem 10. Let G and H be finite groups. If V is an irreducible representation of G and W is an irreducible representation of H, show that $V \otimes W$ is an irreducible representation of $G \times H$: that is, define an action of $G \times H$ on $V \otimes W$ and show that the resulting representation is irreducible. This is called the *external tensor product*; it is not to be confused with the internal tensor product of Definition 1. Show that every irreducible representation of $G \times H$ is of this form. (Hint: use character theory.)

Problem 11. Prove *directly* that the induced representation of Definition 17b) is unique up to isomorphism. That is, fix two sets of coset representatives $\sigma_1, \ldots, \sigma_k$ and $\sigma'_1, \ldots, \sigma'_k$ for G/H with $\sigma_i^{-1}\sigma'_i \in H$, and let

$$V = \bigoplus_{i=1}^{k} W_i, \quad W_i = \{\, w_i \mid w \in W \,\}, \quad \text{with } g \cdot w_i := (h \cdot w)_j, \text{ where } g\sigma_i = \sigma_j h,$$
$$V' = \bigoplus_{i=1}^{k} W'_i, \quad W'_i = \{\, w'_i \mid w \in W \,\}, \quad \text{with } g \cdot w'_i := (h' \cdot w)_j, \text{ where } g\sigma'_i = \sigma'_j h'.$$

Give an explicit G-isomorphism $V \to V'$.

Problem 12. There is another definition of the induced representation that is more natural in some settings. Fix an H-representation (ρ, W), and let $\operatorname{Ind}_H^G \rho$ be the induced representation of Definition 17b), attached to a fixed set $\sigma_1, \ldots, \sigma_k$ of left coset representatives for G/H. Let

$$\operatorname{ind}_H^G \rho = \{\, f\colon G \to W \mid f(hg) = \rho(h) f(g) \text{ for all } h \in H,\ g \in G \,\},$$

with G-action given by $g \cdot f(g') := f(g'g)$.

a) Prove that $\operatorname{ind}_H^G \rho$ is a G-representation. Specifically, prove that $g \cdot f(g') := f(g'g)$ is a G-action on the space of functions $G \to W$, and that if $f \in \operatorname{ind}_H^G \rho$ then so is $g \cdot f$.
b) Prove that $\operatorname{Ind}_H^G \rho \cong \operatorname{ind}_H^G \rho$ by showing that

$$w_i \longmapsto f_{w,i}, \qquad \text{where} \qquad \underset{(h\in H)}{f_{w,i}\big(h\sigma_j^{-1}\big)} = \begin{cases} h \cdot w, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

is a G-isomorphism.

Problem 13. A *virtual representation* over $\mathbb{C}$ of a finite group G is a formal integer linear combination of irreducible G-representations; if all coefficients are nonnegative, this is an honest G-representation. Likewise, a *virtual character* is an element of $\mathbb{C}_{\mathrm{cl}}(G)$ with integer coefficients in the basis of irreducible characters.

a) If χ is a virtual character and $(\chi, \chi) = 1$, prove that exactly one of $\pm\chi$ is an irreducible character.
b) In the setting of Section 8, fix a one-dimensional representation $\nu \colon L^\times \to \mathbb{C}$, let $\alpha = \nu|_{k^\times}$, and let ρ_ν be the virtual representation

$$\rho_\nu := \widetilde{\rho}_1 \otimes \rho_{\alpha,1} - \rho_{\alpha,1} - \operatorname{Ind}_{L^\times}^{G} \nu.$$

Compute the virtual character of ρ_ν, and prove that it is irreducible.

Problem 14. Again in the setting of Section 8, recall the Weil group $W := L^\times \rtimes \langle\sigma\rangle$, where $\sigma^2 = 1$ and $\sigma\ell = \overline{\ell}\,\sigma$ (σ can be thought of as the nontrivial automorphism of L fixing k). Let τ be a two-dimensional representation of W, and write $\tau|_{L^\times} = \nu_1 \oplus \nu_2$.

a) If τ is irreducible, show that $\overline{\nu_1} = \nu_2$, and that neither ν_1 nor ν_2 factors through the norm map $N_{L/k}$.
b) If τ is reducible, show that ν_1 and ν_2 factor through $N_{L/k}$.

PART II

Representations of the symmetric groups

9. Partitions, tableaux, and Young subgroups

Let S_n be the symmetric group on n letters.

The conjugacy classes of S_n are parametrized by (integer) *partitions*

$$\lambda = (\lambda_1, \dots, \lambda_k) \vdash n,$$

which are sequences of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ (the "parts") such that $\lambda_1 + \cdots + \lambda_k = n$.

$w \in S_n$ has *cycle type* λ if λ_i is the size of the ith largest cycle in w for all i.

We want one S_n-irrep for every partition $\lambda \vdash n$.

The *Young diagram* (or *Ferrers diagram*) for λ is the top-and-left justified array of boxes with λ_i boxes in row i.

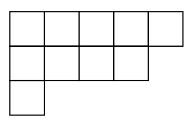
The *length* of λ is the number $\ell(\lambda)$ of parts of λ, i.e. the number of rows in the Young diagram for λ.

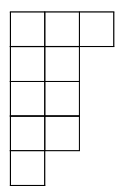
The *size* (or *order*) of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$, i.e. the number of boxes in the Young diagram for λ.

The *conjugate partition* of λ is the partition λ' corresponding to the Young diagram of λ reflected over the line $y = -x$.

The number of columns of the Young diagram for λ equals $\lambda_1 = \ell(\lambda')$, and $|\lambda'| = |\lambda|$.

For example,

$\lambda = (5, 4, 1)$  $\ell(\lambda) = 3,$

$\lambda' = (3, 2, 2, 2, 1)$  $\ell(\lambda') = 5,$

and $|\lambda| = |\lambda'| = 10$.

Three orders on partitions:

- *Containment* (Young's lattice): $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i. (partial order)
- *Dominance order*: $\lambda \trianglelefteq \mu$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all i. (partial order)
- *Lexicographic order*: $\lambda \leq \mu$ if $\lambda = \mu$, or $\lambda_i < \mu_i$ for the minimal i such that $\lambda_i \neq \mu_i$. (total order)

Each refines the one above it.

Definition 20. A *(Young) tableau* T of *shape* λ is a filling of the boxes of (the Young diagram of) λ with positive integers.

- T is *semistandard* if the entries increase weakly along rows and strictly down columns.
- T is *standard* if it is semistandard and has entries $1, 2, \dots, |\lambda|$, appearing once each.

For example,

$$\begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 3 & 5 & 6 \\ \hline 4 & 5 & 7 & 7 \\ \cline{1-4} 5 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 6 & 8 \\ \hline 3 & 5 & 7 & 10 \\ \cline{1-4} 9 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline \color{red}{0} & 1 & 4 & \color{red}{2} & 2 \\ \hline \color{red}{3} & 5 & 7 & 7 \\ \cline{1-4} \color{red}{3} \\ \cline{1-1} \end{array}$$

semistandard standard neither

Lemma 21 (Dominance Lemma). *Let $\lambda, \mu \vdash n$, let T be a tableau of shape λ and S a tableau of shape μ, each with entries $1, \dots, n$ appearing exactly once. If the entries in row i of T all live in different columns of S for all i, then $\lambda \trianglelefteq \mu$.*

Proof. Sort the columns of S so that the entries in the first i rows of T all appear in the first i rows of (the sorted) S. Then

$$\begin{aligned} \lambda_1 + \cdots + \lambda_i &= \#\text{entries in the first } i \text{ rows of } T \\ &\leq \#\text{entries in the first } i \text{ rows of } S \\ &= \mu_1 + \cdots + \mu_i. \end{aligned}$$ □

Definition 22. The *Young subgroup* (or *parabolic subgroup*) of S_n corresponding to a partition $\lambda \vdash n$ is

$$\begin{aligned} S_\lambda &:= S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k} \\ &= \underbrace{S_{\{1,\dots,\lambda_1\}}}_{\substack{\text{permutes first} \\ \lambda_1 \text{ integers}}} \times \underbrace{S_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}}}_{\substack{\text{permutes next} \\ \lambda_2 \text{ integers}}} \times \cdots \times \underbrace{S_{\{n-\lambda_k+1,\dots,n\}}}_{\substack{\text{permutes last} \\ \lambda_k \text{ integers}}}. \end{aligned}$$

More generally, let T be a standard tableau of shape λ. The *row stabilizer* of T is the subgroup

$$R_T := \{\, w \in S_n \mid w \text{ preserves the rows of } T \,\} \subseteq S_n,$$

and the *column stabilizer* of T is the subgroup

$$C_T := \{\, w \in S_n \mid w \text{ preserves the columns of } T \,\} \subseteq S_n.$$

We have $R_T \cong S_\lambda$ and $C_T \cong S_{\lambda'}$.

Example 10. For

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 6 & 8 \\ \hline 3 & 5 & 7 & 10 \\ \cline{1-4} 9 \\ \cline{1-1} \end{array}$$

we have

$$\begin{aligned} R_T &= S_{\{1,2,4,6,8\}} \times S_{\{3,5,7,10\}} \times S_{\{9\}} \;\cong\; S_{(5,4,1)}, \\ C_T &= S_{\{1,3,9\}} \times S_{\{2,5\}} \times S_{\{4,7\}} \times S_{\{6,10\}} \times S_{\{8\}} \;\cong\; S_{(3,2,2,2,1)}. \end{aligned}$$

Definition 23. For any finite group G, the *group algebra* $\mathbb{C}[G]$ is the vector space with basis indexed by the elements of G and multiplication induced from G.

For example, $\mathbb{C}[S_3] = \big\langle (), (12), (13), (23), (123), (132) \big\rangle$, and

$$\big[() + (13)\big]\big[(123) - 3(12)\big] = (123) - 3(12) + (23) - 3(123).$$

(Note that the G-action on $\mathbb{C}[G]$ is the regular representation.) Any G-representation can be extended to a $\mathbb{C}[G]$-module by linearity:

$$\rho(ag + bh) := a\rho(g) + b\rho(h).$$

For example, if

$$\rho\big((12)\big) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \rho\big((123)\big) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$

then

$$\rho\big((12) + (123)\big) = \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}.$$

10. Tabloids, polytabloids, and Specht modules

Let T be any tableau of shape λ with entries (exactly) $1, 2, \dots, n$ (not necessarily standard). Recall the row and column stabilizers:

$$\begin{aligned} R_T &:= \{\, w \in S_n \mid w \text{ preserves the rows of } T \,\}, \\ C_T &:= \{\, w \in S_n \mid w \text{ preserves the cols. of } T \,\}. \end{aligned}$$

Definition 24. Call two tableaux T, T' of the same shape λ *(row) equivalent*, $T \sim T'$, if $T' \in R_T T$.
A (λ-)*tabloid* is an equivalence class

$$\{T\} := R_T T = \big\{\, T' \;\big|\; T' \sim T \,\big\} \qquad \text{(“tableaux with unordered row entries”)}$$

For example,

$$\left\{ \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \right\} = \begin{array}{l} \hline 1\ 4\ 5 \\ \hline 2\ 3 \\ \hline \end{array}$$

There is an S_n-action on the set of λ-tabloids given by $w \cdot \{T\} := \{wT\}$. We define M^λ to be the S_n-permutation representation associated to this action.

Claim. This action is well-defined.

Proof. If $T \sim T'$, we need to show that $wT \sim wT'$, so that $\{wT\} = \{wT'\}$. Let $\sigma T = T'$ with $\sigma \in R_T$.

(e.g. with $w = (13)$:)

$$T = \begin{array}{l} \hline 1\ 2 \\ \hline 3 \\ \hline \end{array} \sim \begin{array}{l} \hline 2\ 1 \\ \hline 3 \\ \hline \end{array} = T', \qquad wT = \begin{array}{l} \hline 3\ 2 \\ \hline 1 \\ \hline \end{array} \sim \begin{array}{l} \hline 2\ 3 \\ \hline 1 \\ \hline \end{array} = wT'.$$

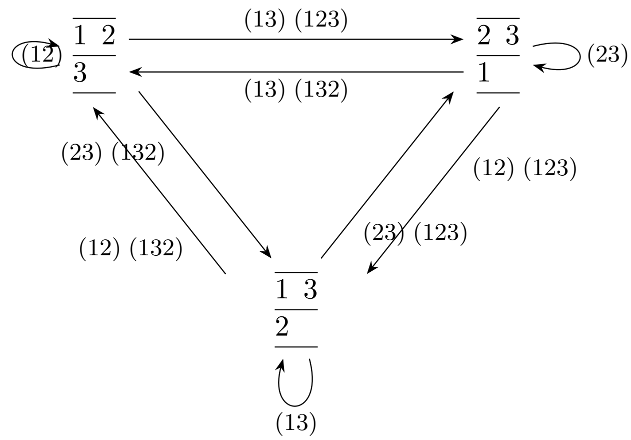
Then $R_{wT} = wR_T w^{-1}$, so $w\sigma w^{-1} \in R_{wT}$, and $w\sigma w^{-1}(wT) = w\sigma T = wT'$, so $wT \sim wT'$. $\square$

Example 11.

a) $\lambda = \begin{array}{|c|c|} \hline & \\ \hline \\ \cline{1-1} \end{array}$, and

$$M^\lambda = \mathbb{C}\left[\begin{array}{l} \hline 1\ 2 \\ \hline 3 \\ \hline \end{array}, \begin{array}{l} \hline 1\ 3 \\ \hline 2 \\ \hline \end{array}, \begin{array}{l} \hline 2\ 3 \\ \hline 1 \\ \hline \end{array} \right]$$

with the S_3-action



b) $M^{(n)}$ is the trivial representation.

$$\begin{array}{l}\hline 1\ 2\ \cdots\ n \\ \hline\end{array}$$

c) $M^{(1^n)}$ is the regular representation.

$$\begin{array}{c}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline\end{array}$$

d) $M^{(n-1,1)}$ is the permutation representation of S_n on $\mathbb{C}^n$.

$$\begin{array}{l}\hline 1\ 2\ \cdots\ (a-1)\ \ (a+1)\ \ \cdots\ n \\ \hline a \\ \hline\end{array}$$

Notice that this is not irreducible: it has the S_n-invariant subspace spanned by

$$\sum_a \begin{array}{c}\hline \cdots \\ \hline a \\ \hline\end{array}$$

Definition 25. A G-representation V is *cyclic* if there exists $v \in V$ such that

$$V = \mathbb{C}[G]\, v.$$

We say V is *generated* by v.

Remark 3. V is irreducible $\iff$ V is generated by v for all $v \in V$.

Proposition 26. *M^λ is cyclic, and all tabloids are generators. We have* $\dim M^\lambda = \dfrac{n!}{\lambda!}$ *where* $\lambda! = \lambda_1!\,\lambda_2!\cdots\lambda_{\ell(\lambda)}!$.

Proof. Since S_n acts transitively on $1, \ldots, n$, it acts transitively on all tableaux — hence, tabloids — with these entries. We can get from $\{T\}$ to any basis element of M^λ, and thus to any linear combination of these basis elements. The last sentence is because $|R_T| = \lambda!$. $\square$

Definition 27. Let

$$\kappa_T := \sum_{w \in C_T} (-1)^w w \;\in\; \mathbb{C}[S_n] \;=\; \kappa_{C_1}\kappa_{C_2}\cdots\kappa_{C_k},$$

where $C_1, C_2, \dots, C_k$ are the columns of T. The *polytabloid* associated to T is

$$e_T := \kappa_T\{T\}.$$

Remark 4. e_T depends on T, not just $\{T\}$. For example, with

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \qquad T' = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \qquad \{T\} = \begin{array}{l}\hline 1\ 2 \\ \hline 3 \\ \hline\end{array} = \{T'\},$$

we have $\kappa_T = () - (13)$ and $\kappa_{T'} = () - (23)$, so

$$e_T = \begin{array}{l}\hline 1\ 2 \\ \hline 3 \\ \hline\end{array} - \begin{array}{l}\hline 2\ 3 \\ \hline 1 \\ \hline\end{array}, \qquad e_{T'} = \begin{array}{l}\hline 1\ 2 \\ \hline 3 \\ \hline\end{array} - \begin{array}{l}\hline 1\ 3 \\ \hline 2 \\ \hline\end{array}.$$

Definition 28. The *Specht module* S^λ is the submodule of M^λ spanned by polytabloids.

Proposition 29. *S^λ is a cyclic S_n-module, generated by any polytabloid.*

Proof. We prove this by showing that $we_T = e_{wT}$. Using Definition 27,

$$\begin{aligned}
e_{wT} = \kappa_{wT}\{wT\} &= \sum_{u \in C_{wT}} (-1)^u\, u\{wT\} \\
&= \sum_{u \in wC_Tw^{-1}} (-1)^u\, uw\{T\} \\
&= \sum_{u' \in C_T} (-1)^{u'}\, wu'\{T\} && (u = wu'w^{-1}) \\
&= w\kappa_T\{T\} \\
&= we_T.
\end{aligned}$$

□

11. Irreducibility of the Specht modules

Example 12.

a) $\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$. Then

$$\begin{aligned}
e_{\begin{smallmatrix}1&2\\3\end{smallmatrix}} &= \begin{array}{l}\hline 1\ 2 \\ \hline 3 \\ \hline\end{array} - \begin{array}{l}\hline 2\ 3 \\ \hline 1 \\ \hline\end{array} = -\, e_{\begin{smallmatrix}3&2\\1\end{smallmatrix}}, \\
e_{\begin{smallmatrix}1&3\\2\end{smallmatrix}} &= \begin{array}{l}\hline 1\ 3 \\ \hline 2 \\ \hline\end{array} - \begin{array}{l}\hline 2\ 3 \\ \hline 1 \\ \hline\end{array} = -\, e_{\begin{smallmatrix}2&3\\1\end{smallmatrix}}, \\
e_{\begin{smallmatrix}2&1\\3\end{smallmatrix}} &= \begin{array}{l}\hline 1\ 2 \\ \hline 3 \\ \hline\end{array} - \begin{array}{l}\hline 1\ 3 \\ \hline 2 \\ \hline\end{array} = -\, e_{\begin{smallmatrix}3&1\\2\end{smallmatrix}},
\end{aligned}$$

and $e_{\substack{3\,1\\2}} = e_{\substack{1\,2\\3}} + e_{\substack{1\,3\\2}}$. So

$$S^\lambda = \mathbb{C}\big[e_{\substack{1\,2\\3}},\, e_{\substack{1\,3\\2}}\big].$$

b) $S^{(n)} = M^{(n)}$ is the trivial representation.

c) $S^{(1^n)}$ is the sign representation, since if $T = \begin{array}{|c|}\hline a_1\\\hline a_2\\\hline \vdots\\\hline a_n\\\hline\end{array}$, then

$$e_T = \sum_{w \in S_n} (-1)^w \left(\begin{array}{c}\hline a_{w(1)}\\\hline a_{w(2)}\\\hline \vdots\\\hline a_{w(n)}\\\hline\end{array}\right) = \pm\, e_{\substack{1\\2\\\vdots\\n}}.$$

d) $S^{(n-1,1)}$ is the submodule of $M^{(n-1,1)}$ spanned by $\{e_{ik},\, i<k\}$ where

$$e_{ik} := e_{\substack{i\,\cdots\\k}} = \begin{array}{l}\hline 1\ \cdots\ (k-1)\ \ (k+1)\ \cdots\\\hline k\\\hline\end{array} - \begin{array}{l}\hline 1\ \cdots\ (i-1)\ \ (i+1)\ \cdots\\\hline i\\\hline\end{array}.$$

This is the *reflection representation* of S_n. We have $S^{(n-1,1)} \oplus \text{triv. repn} = M^{(n-1,1)}$.

We will use the S_n-invariant inner product on M^λ:

$$\big\langle \{T\}, \{T'\} \big\rangle := \delta_{\{T\},\{T'\}}.$$

Theorem 30 (Submodule Theorem)**.**

a) Let U be a submodule of M^μ. Then $U \supseteq S^\mu$ or $U \subseteq (S^\mu)^\perp$.

b) S^μ is irreducible.

Remark 5. b) follows immediately from a), since $S^\mu \cap (S^\mu)^\perp = \{0\}$, so if U is irreducible, either $U = S^\mu$ or $U \cap S^\mu = \{0\}$.

However, a) actually holds over any field! When working in characteristic p, $S^\mu \cap (S^\mu)^\perp$ may be nonzero, and the $S^\mu/\big(S^\mu \cap (S^\mu)^\perp\big)$ form a complete set of irreps. (See James [JK81].)

Lemma 31. *Let $u \in M^\mu$, and let T be a tableau with shape λ.*

a) If $\kappa_T u \neq 0$, then $\lambda \trianglerighteq \mu$.

b) If $\lambda = \mu$, then $\kappa_T u$ is a multiple of e_T.

Proof. u is a linear combination of μ-tabloids, so we can reduce to the case where $u = \{S\}$ for some μ-tableau S, and extend by linearity.

a) Suppose $\lambda \not\trianglerighteq \mu$. By the Dominance Lemma (Lemma 21), there exist two entries i, j in the same row of S that appear in the same column of T. We have

$$\kappa_T = \sum_{w \in C_T} (-1)^w w = \left[\sum_{\substack{w \in C_T\\ w(i) < w(j)}} (-1)^w w\right] \big(() - (i,j)\big),$$

and since $(i,j)\{S\} = \{S\} = ()\{S\}$, we get $\kappa_T\{S\} = 0$.
b) If there exist two entries i, j in the same row of S that appear in the same column of T, the argument for part a) shows that $\kappa_T\{S\} = 0$. Otherwise, we can permute each column of T and obtain a tableau which is row equivalent to S (proof: look at the first column of T, and proceed by induction), i.e. there exists $\sigma \in C_T$ such that $\sigma\{T\} = \{S\}$. Then

$$\begin{aligned}\kappa_T\{S\} = \kappa_T\sigma\{T\} &= \sum_{w\in C_T}(-1)^w w\sigma\{T\} \\ &= \pm\sum_{w\in C_T}(-1)^{w\sigma} w\sigma\{T\} \\ &= \pm\sum_{w'\in C_T}(-1)^{w'} w'\{T\} \\ &= \pm\,\kappa_T\{T\} = \pm\,e_T. \end{aligned}$$ □

Proof of the Submodule Theorem. Let $u \in U$, and let T be a μ-tableau. By Lemma 31, $\kappa_T u = f e_T$ for some $f \in \mathbb{C}$. Since U is S_n-invariant, this means $f e_T \in U$.

If for any choice of u and T, $f \neq 0$, then $e_T \in U$, so since e_T generates S^μ, $S^\mu \subseteq U$.

Otherwise, $\kappa_T u = 0$ for all u, T. We have

$$\begin{aligned}\langle u, e_T\rangle &= \langle u, \kappa_T\{T\}\rangle \\ &= \sum_{w\in C_T}(-1)^w\langle u, w\{T\}\rangle \\ &= \sum_{w^{-1}\in C_T}(-1)^w\langle u, w^{-1}\{T\}\rangle && \text{(inverting } w\text{)} \\ &= \sum_{w\in C_T}(-1)^w\langle wu, \{T\}\rangle && \text{(by } S_n \text{ invariance)} \\ &= \langle \kappa_T u, \{T\}\rangle = 0,\end{aligned}$$

so $u \in (S^\mu)^\perp$ for all $u \in U$. □

Theorem 32 (Decomposition Triangularity Theorem)**.** *The S^λ are mutually inequivalent, and therefore form a complete set of S_n-irreps. M^μ decomposes as*

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda$$

where $m_{\mu,\mu} = 1$.

Proof. Let $\phi \in \mathrm{Hom}_{S_n}(S^\lambda, M^\mu)$. This extends to an S_n-homomorphism $M^\lambda \to M^\mu$ by setting $\phi\big((S^\lambda)^\perp\big) = 0$. We have

$$\phi(e_T) = \phi\big(\kappa_T\{T\}\big) = \kappa_T\phi\big(\{T\}\big),$$

and since $\phi(\{T\})$ is a linear combination of μ-tabloids, by Lemma 31a) this is 0 unless $\lambda \trianglerighteq \mu$.

In particular, since $S^\lambda \subseteq M^\lambda$, if $S^\lambda \cong S^\mu$, then $\mu \trianglerighteq \lambda$ and $\lambda \trianglerighteq \mu$, so $\lambda = \mu$.

If $\lambda = \mu$, by Lemma 31b), $\phi(e_T) = c_T e_T$ for some $c_T \in \mathbb{C}$. However, c_T is independent of T since

$$\phi(e_{wT}) = \phi(we_T) = w\phi(e_T) = w \cdot c_T e_T = c_T e_{wT},$$

so ϕ is multiplication by a scalar, and therefore $\dim \operatorname{Hom}_{S_n}(S^\lambda, M^\mu) = 1$.

By Schur's Lemma, in the decomposition

$$M^\mu = \bigoplus_\lambda m_{\lambda,\mu} S^\lambda,$$

we have $m_{\lambda,\mu} = \dim \operatorname{Hom}_{S_n}(S^\lambda, M^\mu)$, so the above shows that $m_{\mu,\mu} = 1$ and $m_{\lambda,\mu} = 0$ unless $\lambda \trianglerighteq \mu$. $\square$

Next, we want to find a basis for S^λ. Recall that a standard tableau has entries $1, \dots, n$, which increase along rows and down columns.

12. A basis for the Specht modules

Heading towards:

Theorem 33. *The set*

$$E^\lambda = \{\, e_T \mid T \textit{ is a standard tableau of shape } \lambda \,\}$$

is a basis for S^λ.

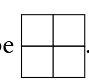
A *composition* is a sequence of nonnegative integers

$$\lambda = (\lambda_1, \dots, \lambda_k) \vDash n, \qquad \text{such that } \lambda_1 + \dots + \lambda_k = n.$$

For any tabloid T of shape λ, let $\{T^i\}$ be the tabloid formed by all elements $\leq i$ in $\{T\}$ (same for polytabloids and (row-increasing) tableaux). This forms a composition

$$\lambda^i := \lambda^i(T) := \operatorname{shape}\big(\{T^i\}\big).$$

We say that $\{S\}$ *dominates* $\{T\}$, written $\{S\} \trianglerighteq \{T\}$, if $\lambda^i(S) \trianglerighteq \lambda^i(T)$ for all i.

Exercise 3. Draw the poset of tabloids of shape .

Lemma 34 (Dominance Lemma for tabloids)**.** *If $k < \ell$ and k appears in a lower row than ℓ in $\{T\}$, then*

$$\{T\} \triangleleft (k, \ell)\{T\} =: \{S\}.$$

Proof. For $i < k$ and $i \geq \ell$ we have $\lambda^i(S) = \lambda^i(T)$. If $k \leq i < \ell$, let r and q be the rows of $\{T\}$ in which k and ℓ appear. Then $\lambda^i(S) = \lambda^i(T)$ with the qth part increased by 1 and the rth part decreased by 1. Since $q < r$, $\lambda^i(S) \trianglerighteq \lambda^i(T)$. $\square$

Proposition 35. *Let T be a standard tableau. If $\{S\}$ appears in e_T, then $\{T\} \trianglerighteq \{S\}$. Furthermore, E^λ is linearly independent.*

Proof. If $\{S\}$ appears in e_T, then $S = \sigma T$ for some $\sigma \in C_T$ and some choice of representative $S \in \{S\}$.

Induction on *column inversions* in S: pairs $k < \ell$ with ℓ in a higher row than k in the same column,

$$\begin{array}{|c|}\hline \ell \\ \hline \vdots \\ \hline k \\ \hline \end{array}$$

By the Dominance Lemma, $S \lhd (k, \ell)\{S\}$.

Since T is standard, it has no column inversions, and σ is a product of column-inversion-reducing transpositions. Thus, the decomposition of the standard polytabloids into tabloids is triangular:

$$\begin{array}{c} \text{all tabloids, ordered by dominance} \\ \begin{array}{ll} & \quad \{T_1\}\cdots\{T_k\} \qquad \cdots \\ \text{std. polytab. ordered by dominance} \left\{ \begin{array}{c} e_{T_1} \\ e_{T_2} \\ \vdots \\ e_{T_k} \end{array} \right. & \begin{bmatrix} 1 & & & \\ & 1 & & * \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \end{array} \end{array}$$

Hence E^λ is linearly independent. $\square$

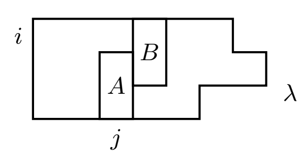
Now for spanning.

Definition 36.

a) Let A and B be disjoint subsets of $[n]$. Fix coset representatives $\sigma_1, \ldots, \sigma_k$ for $S_{A\cup B}/S_A \times S_B$. The corresponding *Garnir element* is

$$g_{A,B} := \sum_{i=1}^{k} (-1)^{\sigma_i} \sigma_i.$$

b) If A and B are these entries of a tableau (adjacent columns, one row overlap, union has all rows)



then we choose $\sigma_1, \ldots, \sigma_k$ such that in $\sigma_i T$ the entries in A and B are increasing down columns.

Exercise 4. Find the Garnir element $g_{A,B}$ for

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 5 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}, \qquad A = \{5, 6\}, \qquad B = \{2, 4\}.$$

The answer is

$$g_{A,B} = () - (45) + (245) + (465) - (2465) + (25)(46),$$

corresponding to

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 5&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&4&3\\\hline 2&5\\\cline{1-2} 6\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&4&3\\\hline 2&6\\\cline{1-2} 5\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|c|}\hline 1&5&3\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

decreasing increasing

Proposition 37. *Under this setup,* $g_{A,B}\, e_T = 0$*.*

If $H \subseteq S_n$, let $\overline{H} := \sum_{w\in H}(-1)^w w$.

Proof. Since $|A \cup B| > \lambda'_j$, for all $w \in C_T$ there exist $a, b \in A \cup B$ such that a and b are in the same row of wT. But then

$$\overline{S_{A\cup B}}\{wT\} = * \underbrace{\big[() - (a,b)\big]\{wT\}}_{0} = 0.$$

Thus $\overline{S_{A\cup B}}\, e_T = 0$ also.

Now,

$$\overline{S_{A\cup B}} = \sum_{w\in S_{A\cup B}} (-1)^w w = \sum_{\sigma\in S_{A\cup B}/S_A\times S_B} (-1)^\sigma \sigma \sum_{w\in S_A\times S_B} (-1)^w w = g_{A,B}\overline{(S_A \times S_B)},$$

so $g_{A,B}\overline{(S_A \times S_B)}\, e_T = 0$, and the result will follow if $\overline{(S_A \times S_B)}\, e_T$ is a nonzero multiple of e_T. Since $S_A \times S_B \subseteq C_T$, if $w \in S_A \times S_B$, then

$$(-1)^w w\, e_T = (-1)^w \kappa_T\{T\} = \kappa_T\{T\}\, e_T,$$

so $\overline{(S_A \times S_B)}\, e_T = |S_A \times S_B|\, e_T$. □

We will use *column tabloids*

$$[T] := C_T T.$$

For example,

$$\left[\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\right] = \left\{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}\right\} = \begin{array}{|c|c|}1&2\\3\end{array}$$

Dominance order is induced from the map $[T] \mapsto \{T'\}$, and similar facts hold (e.g. the dominance lemma).

Proof of Theorem 33. Linear independence is Proposition 35, so it remains to show that span $E^\lambda = S^\lambda$. We induct on the column tabloid dominance order.

Let

$$T_0 = \begin{array}{|c|c|c|c|}\hline 1 & a{+}1 & b{+}1 & \cdots\\\hline 2 & \vdots & \vdots\\\cline{1-3} 3 & b\\\cline{1-2} \vdots\\\cline{1-1} a\\\cline{1-1}\end{array}$$

be the tableau numbered by columns. T_0 is standard and $[T_0] \trianglerighteq [T]$ for all T.

Fix a tableau T. If $S \in T$, then $e_S = \pm e_T$, so we can assume T has increasing columns. Assume $e_S \in \operatorname{span} E^\lambda$ whenever $[S] \trianglerighteq [T]$. If T is standard, done. Otherwise, T has a descent along a row:

$$
\begin{array}{ccccc}
 & a_1 & & b_1 & \\
 & \vdots & & \vdots & B \\
 & & & \wedge & \\
 & a_i & > & b_i & \\
 & \wedge & & \vdots & \\
A & \vdots & & b_q & \\
 & a_p & & &
\end{array}
$$

By Proposition 37, $g_{A,B}\, e_T = 0$, so

$$
e_T = -\sum_{\substack{j \\ \sigma_j \neq 1}} (-1)^{\sigma_j} e_{\sigma_j T}. \tag{4}
$$

Now, each σ_j is the product of permutations of A, permutations of B, and transpositions (a_i, b_j) with $a_i > b_j$. By the dominance lemma for column tabloids, $[\sigma_j T] \triangleright [T]$ for $\sigma_j \neq 1$, so by induction $e_T \in \operatorname{span} E^\lambda$. $\square$

Let f^λ be the number of standard tableaux of shape λ.

Corollary 38.

a) $\dim S^\lambda = f^\lambda$.

b) $\displaystyle\sum_{\lambda \vdash n} (f^\lambda)^2 = n!$.

c) With respect to the basis E^λ, the matrices for the representation S^λ have integer entries. (Young's natural representation)

d) The character table for S_n has integer entries.

Proof. a) $f^\lambda = \left|E^\lambda\right|$.

b) Apply Corollary 14: $|G| = \sum_{V\,:\,\text{irrep}} (\dim V)^2$.

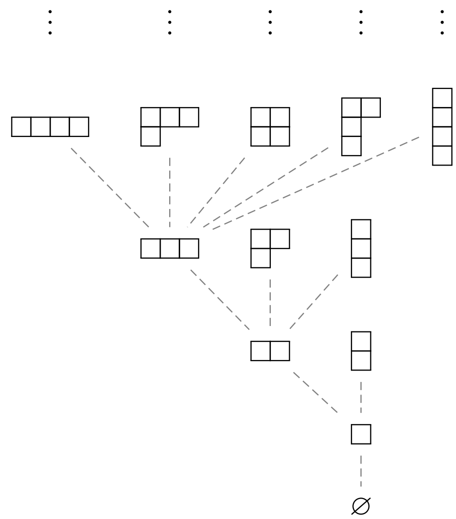
c) $we_T = e_{wT}$, and by (4) each polytabloid is an integer linear combination of standard polytabloids.

d) The trace of an integer matrix is an integer.

$\square$

13. The branching rule

Recall the containment order $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i. This poset is called *Young's lattice* (edges correspond to adding a box):



Let

$$\lambda^+ = \{\, \mu \mid \mu \text{ covers } \lambda \,\}, \qquad \lambda^- = \{\, \mu \mid \lambda \text{ covers } \mu \,\}.$$

Theorem 39 (Branching rule for S_n)**.**

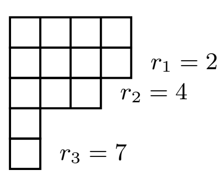
a) $\operatorname{Res}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\mu \in \lambda^-} S^\mu$.

b) $\operatorname{Ind}^{S_n}_{S_{n-1}} S^\lambda = \bigoplus_{\mu \in \lambda^+} S^\mu$.

(multiplicity free!)

By Frobenius reciprocity (Theorem 19), these are equivalent.

Proof. Let $r_1 < r_2 < \cdots < r_k$ denote the corners of λ. Let λ^i be λ without the corner in row r_i. For example,



Let

$$V^{(i)} = \operatorname{span}\left\{ e_T \in E^\lambda \;\middle|\; n \text{ appears in the first } r_i \text{ rows of } T \right\}.$$

We have $\varnothing \subseteq V^{(1)} \subseteq \cdots \subseteq V^{(k)} = S^\lambda$. We claim that the $V^{(i)}$ are subrepresentations, and that $V^{(i)}/V^{(i-1)} \cong S^{\lambda^i}$. Then the result follows since by Maschke's Theorem,

$$S^\lambda \cong V^{(1)} \oplus V^{(2)}/V^{(1)} \oplus \cdots \oplus V^{(k)}/V^{(k-1)}.$$

Let

$$\Theta_i\colon M^\lambda \longrightarrow M^{\lambda^i}, \qquad \{T\} \longmapsto \begin{cases} \{T \setminus n\}, & \text{if } n \text{ is in row } r_i, \\ 0, & \text{otherwise.} \end{cases}$$

Θ_i is S_{n-1}-equivariant and for T standard,

$$\Theta_i(e_T) = \begin{cases} e_{T\setminus n}, & \text{if } n \text{ is in row } r_i, \\ 0, & \text{if } n \text{ is in a higher row,} \end{cases}$$

since if $\{S\}$ appears in e_T, standardness of T means that n appears in a weakly higher row of $\{S\}$. Thus we have $\Theta_i V^{(i)} = S^{\lambda^i}$ and $V^{(i-1)} \subseteq \ker \Theta_i$.

Note that

$$\dim S^\lambda = \sum_{\mu \in \lambda^-} \dim S^\mu \qquad \left(f^\lambda = \sum_{\mu \in \lambda^-} f^\mu \right)$$

since removing n from a standard tableau for λ gives a standard tableau for some S^μ. In the chain

$$\{0\} \subseteq V^{(1)} \cap \ker \Theta_1 \subseteq V^{(1)} \subseteq V^{(2)} \cap \ker \Theta_2 \subseteq V^{(2)} \subseteq \cdots \subseteq V^{(k)} = S^\lambda,$$

we have

$$\dim \frac{V^{(i)}}{V^{(i)} \cap \ker \Theta_i} = \dim \Theta_i V^{(i)} = f^{\lambda^i},$$

so

$$\sum_i \dim \frac{V^{(i)}}{V^{(i)} \cap \ker \Theta_i} = \sum_i f^{\lambda^i} = f^\lambda = \dim S^\lambda,$$

and so we must have $V^{(i)} \cap \ker \Theta_i = V^{(i-1)}$ for all i. Hence $V^{(i)}/V^{(i-1)} \cong S^{\lambda^i}$. □

14. The decomposition of M^μ

Recall Theorem 32:

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda,\mu} S^\lambda, \qquad m_{\mu,\mu} = 1.$$

We will work with tableaux with (potentially) repeated entries. The *content* of T is the composition $a = (a_1, \ldots, a_m)$ where a_i is the number of i's in T. Row and column tabloids are defined similarly to the no-repeats case. For example,

$$\begin{array}{|c|c|c|} \hline 4 & 1 & 4 \\ \hline 1 & 3 \\ \cline{1-2} \end{array} \qquad \begin{array}{l} \text{shape: } \lambda = (3, 2) \\ \text{content: } \mu = (2, 0, 1, 2) \end{array}$$

Recall that T is *semistandard* if entries weakly increase along rows and strictly increase down columns. Slightly reinterpreting the dominance lemma (Lemma 21), if T is semistandard, then $\operatorname{shape}(T) \trianglerighteq \operatorname{content}(T)$.

The *Kostka number* $k_{\lambda,\mu}$ is the number of SSYT of shape λ and content μ.

We will prove:

Theorem 40. $m_{\lambda,\mu} = k_{\lambda,\mu} \qquad$ *(λ, μ partitions).*

Exercise 5. Decompose $M^{(2,2,1)}$ into irreps.

Example 13. $k_{\lambda,1^n} = f^\lambda$, so

$$M^{(1^n)} = \bigoplus_{\lambda \vdash n} f^\lambda S^\lambda \qquad \text{(recall: } M^{(1^n)} \text{ is the regular representation)}$$

Let

$$\begin{aligned} \mathcal{T}_{\lambda\mu} &= \{\text{tableaux of shape } \lambda \text{ and content } \mu\}, \\ \mathcal{T}_{\lambda\mu}^{\mathrm{ss}} &= \{\, T \in \mathcal{T}_{\lambda\mu} \mid T \text{ is semistandard} \,\}\,. \end{aligned}$$

For any shape λ, fix the tableau $t := t_\lambda$

$$t = \begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & \lambda_1 \\ \hline \lambda_1{+}1 & \cdots & \lambda_1{+}\lambda_2 & \\ \cline{1-3} \cdots & & & \\ \cline{1-2} \cdots & |\lambda| & & \\ \cline{1-2} \end{array} \qquad \begin{array}{l} \text{shape } \lambda \\ \text{increasing } 1, \ldots, |\lambda| \\ \text{in English reading order} \end{array}$$

For any λ-tableau T, let $T(i) =$ entry of T in the spot where t_λ has i. For example,

$$T = \begin{array}{|c|c|c|} \hline T(1) & T(2) & T(3) \\ \hline T(4) & T(5) \\ \cline{1-2} \end{array}.$$

Let

$\Theta\colon M^\mu \longrightarrow \mathbb{C}[\mathcal{T}_{\lambda\mu}], \qquad \{S\} \longmapsto T \quad$ where $T(i) =$ the row in which i appears in $\{S\}$.

Let S_n act on $\mathcal{T}_{\lambda\mu}$ via

$$\sigma T(i) := T(\sigma^{-1} i).$$

For example,

$$(124) \cdot \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 3 & 2 \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline 1 & 2 \\ \cline{1-2} \end{array},$$

since

$$(124) \cdot \begin{array}{|c|c|c|} \hline T(1) & T(2) & T(3) \\ \hline T(4) & T(5) \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|} \hline T(4) & T(1) & T(3) \\ \hline T(2) & T(5) \\ \cline{1-2} \end{array}.$$

The proof of Theorem 40 has two steps:

Step A $M^\mu \cong \mathbb{C}[\mathcal{T}_{\lambda\mu}]$.

Step B For all $T \in \mathcal{T}_{\lambda\mu}$, let

$$\Theta_T \colon M^\lambda \longrightarrow \mathbb{C}[\mathcal{T}_{\lambda\mu}] \ (\cong M^\mu), \qquad \{t_\lambda\} \longmapsto \sum_{S \in \{T\}} S \qquad \text{(extended by cyclicity)}$$

Then the maps $\Theta_T|_{S^\lambda}$ form a basis of $\operatorname{Hom}_{S_n}(S^\lambda, M^\mu)$.

Exercise 6. Compute

$$\Theta_T\{t_\lambda\} \qquad \text{and} \qquad \Theta_T\left\{\begin{array}{|c|c|c|}\hline 2 & 4 & 3 \\ \hline 1 & 5 & \\ \cline{1-2}\end{array}\right\} \qquad \text{where} \qquad T = \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 3 & 2 \\ \cline{1-2}\end{array}.$$

Proof of Theorem 40, Step A. Θ is bijective by inspection. It is S_n-equivariant since if $\Theta\{S\} = T$, then

$$\begin{aligned}\sigma\Theta\{S\} = \sigma T(i) = T(\sigma^{-1} i) &= \text{row number of } \sigma^{-1} i \text{ in } \{S\} \\ &= \text{row number of } i \text{ in } \sigma\{S\} \\ &= \Theta\big(\sigma\{S\}\big)(i).\end{aligned}$$

□

Lemma 41 (Dominance lemma for column tabloids with repetition)**.**

a) *Let $k < \ell$. If k appears in a column to the left of ℓ in T, then $[T] \rhd [S]$, where S is the tableau obtained by swapping the k and ℓ in T.*

b) *If T is semistandard and $S \in \{T\}$, then $[T] \trianglerighteq [S]$.*

Proof. Similar to past dominance lemmata. □

Proof of Theorem 40, Step B — linear independence. This has similarities to the proof of Theorem 33, so we sketch it.

Let $\overline{\Theta}_T = \Theta_T|_{S^\lambda}$. First we claim that the elements

$$\left\{\, \overline{\Theta}_T \;\middle|\; T \in \mathcal{T}^{\mathrm{ss}}_{\lambda\mu} \,\right\}$$

are linearly independent. Apply them to e_t:

$$\overline{\Theta}_T e_t = \Theta_T \kappa_t\{t\} \underset{\text{stabilizes cols.}}{=} \kappa_t\, \Theta_T\{t\} = \kappa_t \sum_{S' \in \{T\}} S'.$$

If S appears in this sum, then by Lemma 41b), $[T] \trianglerighteq [S]$. Thus by triangularity, the $\overline{\Theta}_T e_t$ — and hence the $\overline{\Theta}_T$ — are linearly independent. □

Proof of Theorem 40, Step B — spanning. Let $\varphi \in \operatorname{Hom}_{S_n}(S^\lambda, M^\mu)$. Write

$$\varphi e_t = \sum_T c_T\, T \ \in \ \mathbb{C}[\mathcal{T}_{\lambda\mu}],$$

and let

$$L_\varphi = \left\{\, S \in \mathcal{T}^{\mathrm{ss}}_{\lambda\mu} \;\middle|\; [S] \rhd [T] \text{ for some } T \text{ appearing in } \varphi e_t \,\right\}.$$

Induction on $|L_\varphi|$.

- If $|L_\varphi| = 0$, then choose some T appearing in φe_t. If T is not semistandard:

$$\begin{array}{ccccc} & & & b_1 & \\ & & & \wedge & \\ & & & \vdots & B \\ & & & \wedge & \\ & a_i & > & b_i & \\ & \wedge & & & \\ A & \vdots & & & \\ & a_p & & & \end{array}$$

  then applying the Garnir element $g_{A,B}$ to φe_t shows that $[T]$ is not maximal in φe_t; i.e. if $|L_\varphi| = 0$, then $\varphi = 0$.

- Otherwise, choose T appearing in φe_T such that $[T]$ is maximal. Then T is semistandard. Let

$$\varphi' = \varphi - c_T \overline{\Theta}_T,$$

$$\varphi' e_t = \sum_{T'} c_{T'} T' - c_T \sum_{S \in \{T\}} S.$$

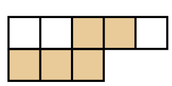
  This subtracts off T and every S with $[S] = [T]$, so $L_{\varphi'} \subseteq L_\varphi$, and by induction φ', and thus φ, are in the span of the $\overline{\Theta}_{T'}$.

□

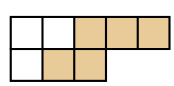
15. Characters of Specht modules

Let χ^λ be the character of S^λ, and let w_μ be a permutation of cycle type μ. The Murnaghan–Nakayama rule below computes $\chi^\lambda(w_\mu)$ by a recursion on λ; Problem 15 gives a quite different route to the same character table.

A *border-strip* or *rim hook* ξ is a contiguous subdiagram of λ such that every cell is on the bottom-right border. Its *size* $|\xi|$ is the number of boxes. Its *height* $h(\xi)$ is the number of rows it occupies, minus 1. For example,



size 5
height 1



size 5
height 2

Let $\mu' = (\mu_2, \mu_3, \ldots, \mu_{\ell(\mu)})$.

Theorem 42 (Murnaghan–Nakayama rule). *For all $\lambda, \mu \vdash n$,*

$$\chi^\lambda(w_\mu) = \sum_\xi (-1)^{h(\xi)} \chi^{\lambda \setminus \xi}(w_{\mu'}),$$

where the sum is over all border strips of λ of size μ_1.

Proof. Deferred to Section 29; the argument uses the symmetric-function machinery of Section 28. □

Exercise 7. Compute $\chi^{(4,4,3)}\big(w_{(5,4,2)}\big)$.

Problems

Throughout, χ^λ is the irreducible character of the Specht module S^λ; C_λ is the conjugacy class of permutations of cycle type λ, and w_λ a representative of it; and S_λ is the Young subgroup $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_{\ell(\lambda)}}$.

Problem 15. In this problem we compute the character table of S_n. Define matrices $A = (A_{\lambda\mu})$ and $B = (B_{\lambda\mu})$ by

$$A_{\lambda\mu} = |S_\lambda \cap C_\mu|\,, \qquad\qquad B_{\lambda\mu} = |S_\mu|\; k_{\lambda\mu},$$

where rows and columns are ordered by lexicographic order, largest first.

a) Explain why $M^\mu = \mathrm{Ind}_{S_\mu}^{S_n} V_{\mathrm{triv}}$. (a sketch is fine for this part)
b) Show that A and B are upper triangular and invertible.
c) Prove that $B(A^\intercal)^{-1}$ is the character table of S_n.
d) Compute A, B, and the character table of S_n when $n = 4$.

Problem 16. Let M be the character table of a group G.

a) Use the orthogonality relations to give a formula, defined up to sign, for $\det M$.
b) For $\lambda \vdash n$, let $m_i(\lambda)$ be the number of parts of size i. Using part a) in the case $G = S_n$, together with Problem 15, prove the remarkable fact that for all n,

$$\prod_{\lambda \vdash n} \prod_{1 \le i \le \ell(\lambda)} \lambda_i = \prod_{\lambda \vdash n} \prod_{i \ge 1} m_i(\lambda)!\ .$$

Problem 17. In this problem we consider Specht modules over an arbitrary field F, as covered in James [JK81]. With respect to this field, M^λ and S^λ are defined the same way as over $\mathbb{C}$ — in particular the coefficient of each tabloid appearing in a polytabloid is ± 1. The Submodule Theorem (Theorem 30) still holds: with respect to the S_n-invariant bilinear form, which is no longer necessarily an inner product, any submodule of M^μ either contains S^μ or is contained in $(S^\mu)^\perp$.

a) Prove that $S^\mu \cap (S^\mu)^\perp$ is a submodule, and that it either equals S^μ or is the unique maximal proper submodule of S^μ. Prove that the quotient module $S^\mu/\big(S^\mu \cap (S^\mu)^\perp\big)$ is zero or irreducible.
b) In the case $\mu = (3,1)$, show that over any field either $S^\mu \cap (S^\mu)^\perp = \{0\}$ or $(S^\mu)^\perp \subseteq S^\mu$. Give a basis of the quotient module $S^\mu/\big(S^\mu \cap (S^\mu)^\perp\big)$.

Problem 18. For a Young tableau T of shape $\lambda \vdash n$ with entries $1, \ldots, n$, recall the row stabilizer

R_T and the column stabilizer C_T, and define the following elements of $\mathbb{C}[S_n]$:

$$a_T = \sum_{w \in R_T} w, \qquad b_T = \sum_{w \in C_T} (-1)^w w, \qquad c_T = b_T\, a_T.$$

Then c_T is the *Young symmetrizer* associated to T. Note that b_T is what we called κ_T in Definition 27.

a) Show that $a_T \sigma = a_T$ for $\sigma \in R_T$, that $b_T w = (-1)^w b_T$ for $w \in C_T$, and that $c_T\{T\} = |R_T|\, e_T$.
b) Define $\phi \colon \mathbb{C}[S_n]c_T \to S^\lambda$ by $\phi(yc_T) = yc_T\{T\}$. Show that ϕ is an isomorphism of S_n-modules, so that $\mathbb{C}[S_n]c_T \cong S^\lambda$.

Problem 19. The *Hecke algebra* $\mathcal{H} := \mathcal{H}_q(S_n)$ is the $\mathbb{C}$-algebra with generators $T_1, T_2, \ldots, T_{n-1}$ and relations

$$T_i^2 = (q-1)T_i + q, \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad T_i T_j = T_j T_i \text{ when } |i-j| \geq 2.$$

a) When $q = 1$, exhibit an isomorphism between $\mathcal{H}$ and the group algebra $\mathbb{C}[S_n]$. (no need to show bijectivity explicitly, since that can be tricky)
b) A representation of an algebra A is a $\mathbb{C}$-algebra homomorphism $A \to \mathrm{End}(V)$ for some vector space V, sending 1 to the identity matrix. Determine the one-dimensional representations of $\mathcal{H}$.

Problem 20. The *Durfee square* of a partition λ is the largest square that fits inside its diagram; that is, λ has a Durfee square of side length $\geq s$ if and only if $\lambda_s \geq s$. Prove that if λ has a Durfee square of size s and $\ell(\mu) < s$, then $\chi^\lambda(w_\mu) = 0$.

PART III

Lie groups and Lie algebras

16. $\mathfrak{sl}_2(\mathbb{C})$ and its representations

This part turns from groups to Lie algebras and Lie groups. No Lie theory background is assumed; classification results are stated where they are needed, and the main references are Bump [Bum04], Humphreys [Hum72] and Fulton–Harris [FH91]. The aim is the greatest hits (highest-weight representations, complete reducibility). The symmetric group will show up again, in two distinct ways.

Definition 43.

- The *general linear Lie algebra* $\mathfrak{gl}_n := \mathfrak{gl}_n(\mathbb{C})$ is the vector space of $n \times n$ complex matrices $\mathrm{Mat}_n(\mathbb{C})$.
- The *special linear Lie algebra* is the subspace $\mathfrak{sl}_n := \mathfrak{sl}_n(\mathbb{C}) \subseteq \mathfrak{gl}_n$ of matrices with trace 0.
- The *Lie bracket* for $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{sl}_n$ is the commutator
$$[A, B] := AB - BA.$$
- A *Lie algebra representation* is a linear map $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ such that
$$\rho\big([A, B]\big) = \big[\rho(A), \rho(B)\big].$$

Exercise 8. Why is the bracket $[\cdot,\cdot]$, rather than matrix multiplication, the right structure to carry on $\mathfrak{gl}_n$?

Our first task: classify the representations of $\mathfrak{sl}_2$. A basis is

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

and

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Example 14.

a) *Standard representation*: $V = \mathbb{C}^2 = \mathrm{span}\left(x = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\ y = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)$.

$$\begin{aligned} hx &= x & hy &= -y \\ ex &= 0 & ey &= x \\ fx &= y & fy &= 0 \end{aligned}$$

$$\begin{tikzcd} \underset{y}{\overset{-1}{\bullet}} \arrow[r, bend left, "e"] & \underset{x}{\overset{1}{\bullet}} \arrow[l, bend left, "f"] \end{tikzcd}$$

b) *Adjoint representation*: $V = \mathfrak{sl}_2 \cong \mathbb{C}^3$,

$$\mathrm{ad}\colon \mathfrak{g} \longrightarrow \underbrace{\mathfrak{gl}(\mathfrak{g})}_{\text{vector space}}, \qquad X \longmapsto \underbrace{\mathrm{ad}(X)}_{\in \mathfrak{gl}(\mathfrak{g})} \quad \text{where } \mathrm{ad}(X)(Y) := [X, Y].$$

$$\begin{array}{lll} \mathrm{ad}(h)(h) = 0 & \mathrm{ad}(h)(e) = 2e & \mathrm{ad}(h)(f) = -2f \\ \mathrm{ad}(e)(h) = -2e & \mathrm{ad}(e)(e) = 0 & \mathrm{ad}(e)(f) = h \\ \mathrm{ad}(f)(h) = 2f & \mathrm{ad}(f)(e) = -h & \mathrm{ad}(f)(f) = 0 \end{array}$$

$$\begin{tikzcd} \underset{f}{\overset{-2}{\bullet}} \arrow[r, bend left, "e"] & \underset{h}{\overset{0}{\bullet}} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & \underset{e}{\overset{2}{\bullet}} \arrow[l, bend left, "f"] \end{tikzcd}$$

Now let V be any (finite-dimensional) $\mathfrak{sl}_2$-irrep. Then h has some eigenvector $v \in V$ with eigenvalue $\lambda \in \mathbb{C}$. Then

$$hev = ehv + [h,e]v = e\lambda v + 2v = (\lambda+2)ev,$$

and similarly $hfv = (\lambda - 2)fv$.

So $e^i v$, $f^j v$ are linearly independent h-eigenvectors for all i, j. Finite dimensionality shows that eventually these become 0.

Let us take v to be an eigenvector such that $ev = 0$. Choose k maximal such that $f^k v \neq 0$. Then

$$W = \operatorname{span}\left\{v, fv, f^2v, \ldots, f^kv\right\} \leq V.$$

We claim that it is all of V. Clearly $fW \leq W$, and by iterating the Lie bracket,

$$hf^\ell v = (\lambda - 2\ell)f^\ell v \in W \qquad (hv = \lambda v)$$

Finally,

$$\begin{aligned}
efv &= fev + [e,f]v = 0 + hv = \lambda v,\\
ef^2v &= fefv + [e,f]fv = \lambda fv + hfv = (2\lambda - 2)fv,\\
ef^3v &= fef^2v + [e,f]f^2v = (2\lambda-2)f^2v + hf^2v = (3\lambda - 6)f^2v,\\
&\ \ \vdots\\
ef^\ell v &= \big(\ell\lambda - \ell(\ell-1)\big)f^{\ell-1}v.
\end{aligned}$$

So $eW \leq W$, and since V is irreducible, $V = W$.

Last piece: what is λ?

$$0 = ef^{k+1}v = \big((k+1)\lambda - k(k+1)\big)\overbrace{f^kv}^{\neq 0},$$

so $(k+1)\lambda - k(k+1) = 0 \implies \lambda = k$.

We have proved:

Theorem 44. *The irreducible finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ are parametrized by nonnegative integers k. The irrep $V^{(k)}$ has a unique vector v_k (up to scalar multiplication) that satisfies $ev_k = 0$. Furthermore, $V^{(k)}$ has a basis of h-eigenvectors*

$$V^{(k)} = \operatorname{span}\{v_k, v_{k-2}, \ldots, v_{-k}\}$$

such that

$$\begin{aligned}
hv_i &= iv_i\\
ev_{k-2\ell} &= \ell\, v_{k-2\ell+2} \qquad\qquad \Big[\ v_{k-2\ell} := (k-\ell)!\, f^\ell v\ \Big]\\
fv_{k-2\ell} &= (k-\ell)\, v_{k-2\ell-2}
\end{aligned}$$

$$\begin{tikzcd}
v_{-k} \arrow[r, bend left, "e"] & v_{2-k} \arrow[l, bend left, "f"] & \cdots & v_{k-4} \arrow[r, bend left, "e"] & v_{k-2} \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & v_k \arrow[l, bend left, "f"]
\end{tikzcd}$$

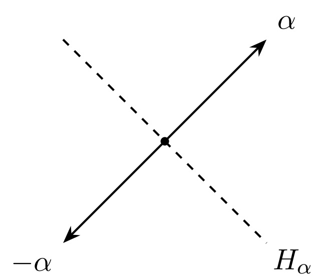
Qualitative features of Theorem 44:

- Basis of h-eigenvectors.
- e "raises" to a higher eigenspace, while f "lowers".
- Representation determined by the largest h-eigenvalue.

Goal: generalize to a broader class of Lie algebras.

17. Root systems

Let E be a finite-dimensional real vector space. For any nonzero $\alpha \in E$, let $s_\alpha \in \mathrm{GL}(E)$ be the reflection across the hyperplane H_α passing through the origin and perpendicular to α.



s_α sends α to $-\alpha$. We have

$$s_\alpha(\beta) = \beta - \underbrace{2\frac{(\beta,\alpha)}{(\alpha,\alpha)}}_{:= \langle \beta, \alpha \rangle} \alpha,$$

where $(\cdot,\cdot)$ is the standard inner product on E.

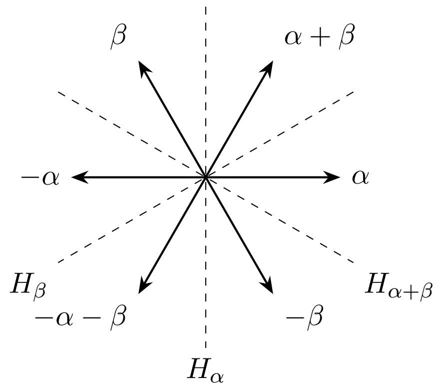
Definition 45. A *root system* Φ is a finite subset of E (whose elements are called *roots*) which satisfies:

a) $\operatorname{span} \Phi = E$.
b) If $\alpha \in \Phi$, then Φ contains $-\alpha$, but no other multiples of α (including 0).
c) For all $\alpha \in \Phi$, s_α preserves Φ.
d) For all $\alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$.
    - projection of β onto $\operatorname{span}(\alpha)$ is a half-integer multiple of α;
    - $s_\alpha(\beta) = \beta$ minus an integer multiple of α.

The *rank* of Φ is $\dim E$.

The *Weyl group* of Φ is the subgroup $W \leq \mathrm{GL}(E)$ generated by the s_α.

Example 15. A_2:



$|\alpha| = |\beta|$

angle between α, β
is 120°

$$\langle\alpha,\alpha\rangle = 2 = \langle\beta,\beta\rangle, \qquad \langle\beta,\alpha\rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = -1 = \langle\alpha,\beta\rangle, \qquad W = \langle s_\alpha, s_\beta, s_{\alpha+\beta}\rangle.$$

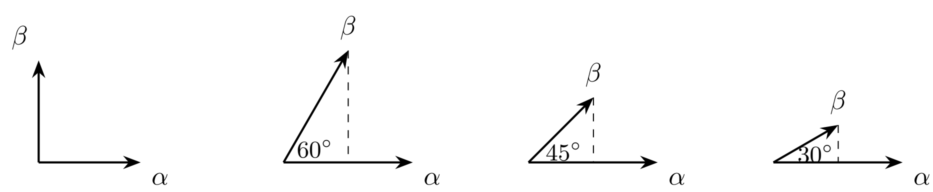
Exercise 9. Determine what these reflections do to the roots, and what relations they satisfy.

What angles θ can we have between roots?

$$\underbrace{\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle}_{\in\mathbb{Z}} = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \cdot 2\frac{(\alpha,\beta)}{(\beta,\beta)} = 4\frac{(\alpha,\beta)^2}{\|\alpha\|^2\,\|\beta\|^2} = (2\cos\theta)^2 \in [0,4]$$

$\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle$	$\cos\theta$	θ	$\frac{\|\alpha\|}{\|\beta\|}$ $(\|\alpha\| \geq \|\beta\|)$
0	0	90°	any
1	$\pm\frac{1}{2}$	60°, 120°	1
2	$\pm\frac{\sqrt{2}}{2}$	45°, 135°	$\sqrt{2}$
3	$\pm\frac{\sqrt{3}}{2}$	30°, 150°	$\sqrt{3}$
4	±1	0°, 180°	1 $(\beta = -\alpha)$

Length ratio:

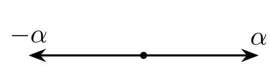


any 1 $\sqrt{2}$ $\sqrt{3}$

We have classified the rank ≤ 2 root systems:

Rank 1:

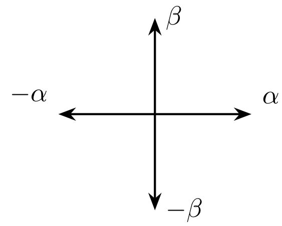
A_1



$s_\alpha^2 = 1$

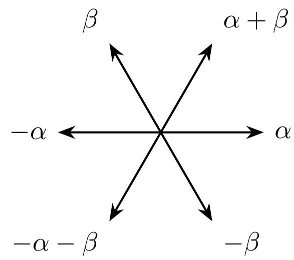
Rank 2:

$A_1 \times A_1$



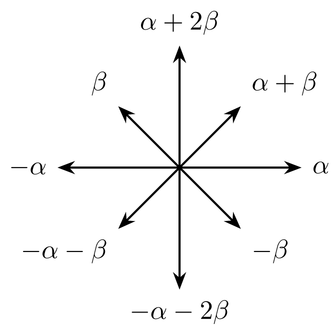
$$s_\alpha^2 = s_\beta^2 = 1$$
$$s_\alpha s_\beta = s_\beta s_\alpha$$

A_2



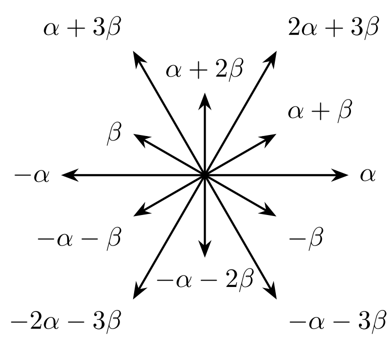
$$s_\alpha^2 = s_\beta^2 = 1$$
$$s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$$

B_2/C_2



$$s_\alpha^2 = s_\beta^2 = 1$$
$$s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha s_\beta s_\alpha$$

G_2



$$s_\alpha^2 = s_\beta^2 = 1$$
$$s_\alpha s_\beta s_\alpha s_\beta s_\alpha s_\beta = s_\beta s_\alpha s_\beta s_\alpha s_\beta s_\alpha$$

For all these root systems,

$$W = \Big\langle s_\alpha,\ s_\beta,\ s_\alpha^2 = s_\beta^2 = 1,\quad \underbrace{s_\alpha s_\beta \cdots}_{m_{\alpha,\beta}\text{ factors}} = s_\beta s_\alpha \cdots \Big\rangle,$$

where

$$m_{\alpha,\beta} = \frac{180^\circ}{180^\circ - \theta}, \qquad (\theta \text{ is the angle between } \alpha \text{ and } \beta)$$

Definition 46. A *base* is a subset $\Delta \subseteq \Phi$ such that

a) Δ is a basis of E;

b) If $\beta \in \Phi$, then $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$, where $k_\alpha \in \mathbb{Z}$ for all α and the k_α are either all nonnegative or all nonpositive.

Elements of Δ are called *simple roots*.

18. Classification of root systems

Recall the definition of a root system $\Phi \subseteq E$ (Definition 45), its rank $\dim E$ and its Weyl group $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$; and recall that every root system has a choice of simple roots $\Delta \subseteq \Phi$ (Definition 46).

Given Δ, we can write $\Phi = \Phi^+ \sqcup \Phi^-$ where

$$\Phi^+ = \left\{ \beta \in \Phi \,\middle|\, \beta = \textstyle\sum_{\alpha \in \Delta} k_\alpha \alpha \text{ with } k_\alpha \geq 0 \right\} \quad \text{positive roots}$$
$$\Phi^- = \left\{ \beta \in \Phi \,\middle|\, \beta = \textstyle\sum_{\alpha \in \Delta} k_\alpha \alpha \text{ with } k_\alpha \leq 0 \right\} \quad \text{negative roots}$$

The *height* of $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$ is $\sum_{\alpha \in \Delta} k_\alpha$.

Example 16. $\Delta = \{\alpha, \beta\}$ for all the rank 2 examples above.

Theorem 47. *Every root system has a base, which is unique up to the action of the Weyl group.*

Proof. See [Hal15, Thm. 8.14, Thm. 8.20] or [Hum72, pp. 48–51]. □

Properties (see Hall [Hal15] or Humphreys [Hum72] for proofs).

- $(\alpha, \beta) < 0$ for all $\alpha \neq \beta \in \Delta$.
- Every root system has a unique *highest root* (with respect to a choice of simple roots).
- Given Δ, there exists a hyperplane $H \subseteq E$ such that Φ^+ and Φ^- lie on opposite sides of H.
- Let $\alpha, \beta \in \Phi$.
  - If $(\alpha, \beta) < 0$, then $\alpha + \beta$ is a root.
  - If $(\alpha, \beta) > 0$, then $\alpha - \beta$ and $\beta - \alpha$ are roots.
- Every $\beta \in \Phi$ can be written $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_k$, $\alpha_i \in \Delta$, such that each partial sum $\alpha_1 + \cdots + \alpha_j$ is a root.
- W is generated by the set of *simple reflections*
$$S = \{ s_\alpha \mid \alpha \in \Delta \}.$$
- Let $\alpha \in \Delta$. Then s_α permutes $\Phi^+ \setminus \{\alpha\}$.
- A *word* for $w \in W$ is a product
$$w = s_{i_1} \cdots s_{i_k}, \qquad (s_{i_j} \in S).$$
  The word is *reduced* if k is minimal (for w). The *length* $\ell(w)$ of w is this k for a reduced word. With this, we have
$$\ell(w) = \left| \left\{ \alpha \in \Phi^+ \,\middle|\, w \cdot \alpha \notin \Phi^+ \right\} \right|.$$

Classification of root systems.

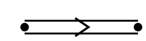
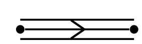
Let Φ be a root system, and fix an ordering $\alpha_1, \ldots, \alpha_n$ of its simple roots. The *Cartan matrix* of Φ is the matrix

$$C = C_\Phi := \big(\langle \alpha_i, \alpha_j \rangle\big)_{ij}.$$

For example,

$$\underset{A_1 \times A_1}{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}} \qquad \underset{A_2}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}} \qquad \underset{B_2/C_2}{\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}} \qquad \underset{G_2}{\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}}$$

The *Dynkin diagram* associated to Φ (or C_Φ, or W) is the graph with n vertices with the following connectivity m_{α_i,α_j}:

$C_{ij}C_{ji}$	m_{ij}	Angle	Length ratio	Dynkin diagram
0	2	90°	?	
1	3	120°	1	
2	4	135°	$\sqrt{2}$	
3	6	150°	$\sqrt{3}$	 longer root

A root system Φ is *irreducible* if it cannot be written $\Phi = \Phi_1 \sqcup \Phi_2$ with $\Phi_2 \subseteq \Phi_1^\perp$; i.e. irreducible $\iff$ connected Dynkin diagram.

Two root systems are *equivalent* if some rotation/scaling sends one to the other, i.e. if they have the same Cartan matrix / Dynkin diagram.

Theorem 48 (Classification of root systems)**.** *Up to equivalence, the irreducible root systems are the classical types A_n, B_n, C_n, D_n and the exceptional types E_6, E_7, E_8, F_4, G_2 listed below.*

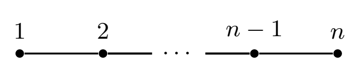
Proof. See Hall [Hal15] or Humphreys [Hum72]. ☐

Classical types:

A_n $(n \geq 1)$:

$$\Phi = \{\, e_i - e_j \mid 1 \leq i, j \leq n+1,\ i \neq j \,\}$$
$$E = \operatorname{span} \Phi = \left\{\, v \in \mathbb{R}^{n+1} \;\middle|\; \text{sum of coefficients} = 0 \,\right\}$$
$$\Delta = \{\, e_i - e_{i+1} \mid 1 \leq i \leq n \,\}$$

$$C_\Phi = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

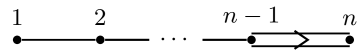


$$W \cong S_{n+1}$$

B_n $(n \geq 2)$:

$$\Phi = \{\, \pm e_i \pm e_j,\ \pm e_i \mid i \neq j \leq n \,\}$$
$$\Delta = \{\, e_i - e_{i+1} \mid 1 \leq i \leq n-1 \,\} \cup \{e_n\}$$

$$C_\Phi = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -2 & 2 \end{bmatrix}$$

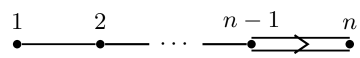


$$W \cong \{\text{signed permutations of } 1, \dots, n\}$$

C_n $(n \geq 2)$, with $C_2 \cong B_2$:

$$\Phi = \{\, \pm e_i \pm e_j,\ \pm 2e_i \mid i \neq j \leq n \,\}$$
$$\Delta = \{\, e_i - e_{i+1} \mid 1 \leq i \leq n-1 \,\} \cup \{2e_n\}$$

$$C_\Phi = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -2 \\ & & & -1 & 2 \end{bmatrix}$$

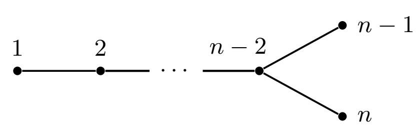


$$W \cong \{\text{signed permutations of } 1, \dots, n\}$$

D_n $(n \geq 4)$:

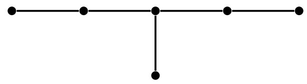
$$\Phi = \{\, \pm e_i \pm e_j \mid i \neq j \leq n \,\}$$
$$\Delta = \{\, e_i - e_{i+1} \mid 1 \leq i \leq n-1 \,\} \cup \{e_{n-1} + e_n\}$$

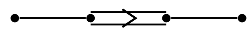
$$C_\Phi = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & \ddots & \ddots & -1 & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{bmatrix}$$

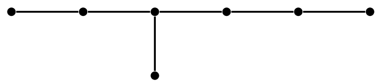


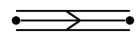
$$W \cong \left\{ \begin{matrix} \text{signed permutations of } 1, \dots, n \\ \text{with an even number of signs flipped} \end{matrix} \right\}$$

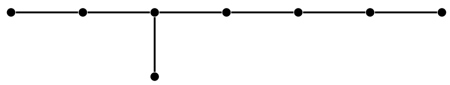
Exceptional types:

E_6 

F_4 

E_7 

G_2 

E_8 

19. Semisimple Lie algebras and the root space decomposition

Recall:

a) $\mathfrak{sl}_2(\mathbb{C})$-irreps
   - determined by their largest h-eigenvalue k ($h = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$);
   - k a nonnegative integer, dimension of the irrep is $k+1$;
   - e/f raise/lower into higher/lower h-eigenspaces.

b) root systems
   - finite subsets of Euclidean space closed under a reflection group ("Weyl group") and satisfying integrality conditions;
   - lots of structure (simple roots, positive roots, etc.);
   - Dynkin classification: A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2.

Definition 49. A (finite-dimensional, complex) *Lie algebra* is a $\mathbb{C}$-vector space $\mathfrak{g}$ with a map $[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ such that

a) $[\cdot,\cdot]$ is bilinear;

b) *skew-symmetry*: $[X,Y] = -[Y,X]$ for all $X, Y \in \mathfrak{g}$;

c) *Jacobi identity*: $\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Remark 6.

- Condition b) implies $[X,X] = 0$.
- $[X,Y] = XY - YX$ satisfies these properties; e.g.

$$\begin{aligned}\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] &\\ = X(YZ-ZY) - (YZ-ZY)X &\\ + Y(ZX-XZ) - (ZX-XZ)Y &\\ + Z(XY-YX) - (XY-YX)Z &\\ = 0.&\end{aligned}$$

Definition 50.

a) An *ideal* of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$ (a subspace closed under $[\cdot,\cdot]$) such that $[X,H] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$, $H \in \mathfrak{h}$.

b) $\mathfrak{g}$ is *indecomposable* if its only ideals are $\mathfrak{g}$ and $\{0\}$, and *simple* if it is indecomposable and has $\dim \geq 2$.

c) $\mathfrak{g}$ is *reductive* if it is a direct sum of indecomposable Lie algebras, and *semisimple* if it is a direct sum of simple Lie algebras.

- $\mathfrak{sl}_n$ is semisimple. (running example)
- $\mathfrak{gl}_n$ is reductive.

Definition 51 (Def./Prop.)**.** A *Cartan subalgebra* of $\mathfrak{g}$ is a subspace $\mathfrak{h} \leq \mathfrak{g}$ such that

a) $\mathfrak{h}$ is *abelian*: $[H, H'] = 0$ for all $H, H' \in \mathfrak{h}$;

b) If $\mathfrak{h} \subsetneq \mathfrak{h}' \leq \mathfrak{g}$, then $\mathfrak{h}'$ is not abelian;

c) For all $H \in \mathfrak{h}$, $\operatorname{ad}_H \colon X \mapsto [H, X]$ is diagonalizable. (previously we called this $\operatorname{ad}(H)$)

Every semisimple Lie algebra has a Cartan subalgebra.

The *rank* of $\mathfrak{g}$ is the dimension of any Cartan subalgebra.

Definition 52. Fix a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. A *root* of $\mathfrak{g}$ (relative to $\mathfrak{h}$) is a nonzero linear functional $\alpha \colon \mathfrak{h} \to \mathbb{C}$ such that there exists $X \in \mathfrak{g} \setminus \{0\}$ with

$$\operatorname{ad}_H(X) = \alpha(H)\, X \qquad \text{for all } H \in \mathfrak{h}.$$

The *root space* $\mathfrak{g}_\alpha$ is the set of all such X (plus 0). Let $\Phi = \Phi_{\mathfrak{g}}$ be the set of roots associated to $\mathfrak{g}$.

Roots seem unlikely, since X must be a simultaneous eigenvector for all of $\mathfrak{h}$. However,

Theorem 53 (Root space decomposition)**.** *If $\mathfrak{g}$ is semisimple with Cartan subalgebra $\mathfrak{h}$, then (as a vector space)*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Proof sketch. Since $\mathfrak{h}$ is abelian, so is the set $\{\operatorname{ad}_H \mid H \in \mathfrak{h}\}$, since by the Jacobi identity if $H, H' \in \mathfrak{h}$,

$$\big[H, [H', X]\big] + \underbrace{\big[X, [H, H']\big]}_{0} + \underbrace{\big[H', [X, H]\big]}_{-\left[H', [H, X]\right] \text{ by antisymmetry}} = 0,$$

so $\operatorname{ad}_H \operatorname{ad}_{H'}(X) = \operatorname{ad}_{H'} \operatorname{ad}_H(X)$.

A set of commuting diagonalizable operators is simultaneously diagonalizable (Problem 21). Choose a basis of $\mathfrak{g}$ (including a basis of $\mathfrak{h}$) such that the ad_H are simultaneously diagonalizable, i.e. for all basis elements X, $\operatorname{ad}_H(X) = \alpha(H)X$ for some function $\alpha \colon \mathfrak{h} \to \mathbb{C}$. Since $[\cdot, \cdot]$ is bilinear, α is a functional, i.e. a root. Thus the basis consists of a basis for $\mathfrak{h}$ plus a basis for $\mathfrak{g}_\alpha$ for all $\alpha \in \Phi$. □

Remark 7. We can think of $\mathfrak{h} =: \mathfrak{g}_0$ as the "root space" for the "root" $\alpha = 0$, since $\operatorname{ad}_H(X) = 0$ for all $H, X \in \mathfrak{h}$.

Important properties:

- If $\alpha, \beta \in \Phi \sqcup \{0\}$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. In particular, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$.

Proof. Let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$, $H \in \mathfrak{h}$. Using the Jacobi identity,

$$\begin{aligned}[H,[X,Y]] &= [[H,X],Y] + [X,[H,Y]] \\ &= [\alpha(H)X, Y] + [X, \beta(H)Y] \\ &= (\alpha(H)+\beta(H))[X,Y]. \quad \square\end{aligned}$$

- $\Phi_{\mathfrak{g}}$ is a root system, in the sense of Definition 45 [Hal15, Thm. 6.34].
- All root spaces are one-dimensional [Hal15, Thm. 6.20]. Furthermore, if $\alpha \in \Phi^+$, there exist nonzero elements $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{h}$ such that

$$[h_\alpha, e_\alpha] = 2e_\alpha, \qquad [h_\alpha, f_\alpha] = -2f_\alpha, \qquad [e_\alpha, f_\alpha] = h_\alpha.$$

  This is a copy of $\mathfrak{sl}_2$!

20. Classification of semisimple Lie algebras

Recall the definition of a Lie algebra (Definition 49) and the root space decomposition (Theorem 53): if $\mathfrak{g}$ is semisimple (a direct sum of simples) with Cartan subalgebra $\mathfrak{h}$ (maximal abelian, $\mathrm{ad}_{\mathfrak{h}}$ diagonalizable), then, as a vector space,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \qquad \text{where} \qquad \mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} \mid \mathrm{ad}_H(X) = \alpha(H)X \ \forall H \in \mathfrak{h} \,\}.$$

Properties: Φ is a root system, root spaces are 1-dimensional, and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Each $\alpha \in \Phi^+$ is associated to a copy of $\mathfrak{sl}_2$, span $\{e_\alpha, f_\alpha, h_\alpha\}$.

Example 17. $\mathfrak{g} = \mathfrak{sl}_n = \{\text{traceless } n \times n \text{ matrices}\}$. Cartan subalgebra:

$$\mathfrak{h} = \left\{ \text{diagonal matrices } \underbrace{\begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}}_{H} \;\middle|\; z_1 + \cdots + z_n = 0 \right\},$$

$$\mathfrak{g} = \mathfrak{h} \oplus \operatorname{span}\{\, E_{ij} \mid i \neq j \,\},$$

where E_{ij} is the matrix with a 1 in position (i,j) and zeros elsewhere. Then

$$\begin{aligned}[H, E_{ij}] &= \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} E_{ij} - E_{ij} \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} \\ &= z_i E_{ij} - z_j E_{ij} \\ &= (z_i - z_j) E_{ij} \\ &= \alpha_{ij}(H)\, E_{ij}, \qquad \text{where } \alpha_{ij} \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix} = z_i - z_j.\end{aligned}$$

The α_{ij} are exactly the A_{n-1} roots

$$\{\, e_i - e_j \mid 1 \le i, j \le n,\ i \ne j \,\},$$

interpreting e_i as the functional $e_i\left(\begin{smallmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{smallmatrix}\right) = z_i$. (potentially confusing notation)

If $i < j$ and $\alpha = \alpha_{ij}$, we have

$$e_\alpha = E_{ij}, \qquad f_\alpha = E_{ji}, \qquad h_\alpha = \begin{array}{cc} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} & \begin{matrix} i \\ j \end{matrix} \end{array}$$

i.e. $h_\alpha = E_{ii} - E_{jj}$, with the nonzero entries in rows and columns i and j.

If $n = 3$, we have

$$\Phi = \{\, \overbrace{e_1 - e_2}^{\alpha_1},\ \overbrace{e_1 - e_3}^{\alpha_1+\alpha_2},\ \overbrace{e_2 - e_3}^{\alpha_2},\ e_2 - e_1,\ e_3 - e_1,\ e_3 - e_2 \,\},$$

where $\Delta = \{\alpha_1, \alpha_2\}$ and $\Phi^+ = \{e_1 - e_2,\ e_1 - e_3,\ e_2 - e_3\}$. The root spaces sit inside $\mathfrak{sl}_3$ as

$$\begin{bmatrix} \mathfrak{h} & \mathfrak{g}_{\alpha_1} & \mathfrak{g}_{\alpha_1+\alpha_2} \\ \mathfrak{g}_{-\alpha_1} & \mathfrak{h} & \mathfrak{g}_{\alpha_2} \\ \mathfrak{g}_{-\alpha_1-\alpha_2} & \mathfrak{g}_{-\alpha_2} & \mathfrak{h} \end{bmatrix}$$

$$e_{\alpha_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad f_{\alpha_1} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad h_{\alpha_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$e_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad f_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad h_{\alpha_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Theorem 54 (Killing–Cartan classification)**.** *Complex semisimple Lie algebras are in correspondence with root systems (and therefore Dynkin diagrams).*

Proof. The vector space and bracket structure are determined by the root space decomposition and the other properties. □

Up to taking direct sums, these are:

		dimension
Classical:		
A_n	$\mathfrak{sl}_{n+1}$	$n(n+2)$
B_n	$\mathfrak{so}_{2n+1}$ (special orthogonal)	$2n^2 + n$
C_n	$\mathfrak{sp}_{2n}$ (symplectic)	$2n^2 + n$
D_n	$\mathfrak{so}_{2n}$ $(n > 1)$	$2n^2 - n$
Exceptional: ([FH91, Ch. 22])		
$E_6,\ E_7,\ E_8,\ F_4,\ G_2$		$78,\ 133,\ 248,\ 52,\ 14$

Symplectic Lie algebras (type C).

$$\mathfrak{sp}_{2n} = \left\{ X \in \mathfrak{gl}_{2n} \;\middle|\; JX + X^T J = 0 \right\}, \qquad \text{where} \quad J = \left[\begin{array}{ccc|ccc} & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \\ \hline -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} n \\ \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} n \end{array}$$

Exercise 10. Writing $X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ in $n \times n$ blocks, what does the condition $X \in \mathfrak{sp}_{2n}$ say about A, B, C, D?

(Answer: B and C are symmetric and $D = -A^{\mathsf{T}}$.)
The Cartan subalgebra is

$$\mathfrak{h} = \mathfrak{sp}_{2n} \cap \{\text{diagonal matrices}\} = \left\{ \left[\begin{array}{ccc|ccc} z_1 & & & & & \\ & \ddots & & & 0 & \\ & & z_n & & & \\ \hline & & & -z_1 & & \\ & 0 & & & \ddots & \\ & & & & & -z_n \end{array}\right] \right\},$$

with roots $\{ \pm e_i \pm e_j,\ \pm 2e_i \mid \ \}$, where $e_i(\,\cdot\,) = z_i$ as above. Writing $H_i := E_{ii} - E_{\underline{i}\,\underline{i}}$ and $\underline{i} := i + n$, the embedded $\mathfrak{sl}_2$'s are:

α	e_α	f_α	h_α
$e_i - e_j$	$E_{ij} - E_{\underline{j}\,\underline{i}}$	$E_{ji} - E_{\underline{i}\,\underline{j}}$	$H_i - H_j$
$e_i + e_j$	$E_{i,\underline{j}} + E_{j,\underline{i}}$	$E_{\underline{i},j} + E_{\underline{j},i}$	$H_i + H_j$
$2e_i$	$E_{i,\underline{i}}$	$E_{\underline{i},i}$	H_i

Special orthogonal Lie algebra (type B (odd) and D (even)).

$$\mathfrak{so}_{2n(+1)} = \left\{ X \in \mathfrak{gl}_{2n(+1)} \;\middle|\; SX + X^T S = 0 \right\},$$

$$\text{where} \quad S = \left[\begin{array}{ccc|ccc|c} & & & 1 & & & \\ & 0 & & & \ddots & & 0 \\ & & & & & 1 & \\ \hline 1 & & & & & & \\ & \ddots & & & 0 & & 0 \\ & & 1 & & & & \\ \hline & 0 & & & 0 & & (1) \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} n \\ \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} n \\ \text{for } \mathfrak{so}_{2n+1} \end{array}$$

If

$$X = \left[\begin{array}{c|c|c} A & B & (E) \\ \hline C & D & (F) \\ \hline (G) & (H) & (J) \end{array}\right] \in \mathfrak{so}_{2n} \quad \big(\in \mathfrak{so}_{2n+1}\big),$$

then B, C are skew-symmetric and $D = -A^T$ ($E = -H^T$, $F = -G^T$, $J = 0$).

The Cartan subalgebra is

$$\mathfrak{h} = \mathfrak{so}_{2n(+1)} \cap \{\text{diagonal matrices}\} = \left\{ \left[\begin{array}{ccc|ccc|c} z_1 & & & & & & \\ & \ddots & & & 0 & & 0 \\ & & z_n & & & & \\ \hline & & & -z_1 & & & \\ & 0 & & & \ddots & & 0 \\ & & & & & -z_n & \\ \hline & 0 & & & 0 & & (0) \end{array}\right] \right\},$$

with roots $\{\pm e_i \pm e_j,\ (\pm e_i)\}$, where $e_i(\,\cdot\,) = z_i$. The embedded $\mathfrak{sl}_2$'s are:

α	e_α	f_α	h_α
$e_i - e_j$	$E_{ij} - E_{\underline{j}\,\underline{i}}$	$E_{ji} - E_{\underline{i}\,\underline{j}}$	$H_i - H_j$
$e_i + e_j$	$E_{i,\underline{j}} - E_{j,\underline{i}}$	$E_{\underline{i},j} - E_{\underline{j},i}$	$H_i + H_j$
$(e_i$	$E_{i,2n+1} - E_{2n+1,\underline{i}}$	$E_{\underline{i},2n+1} - E_{2n+1,i}$	$2H_i)$

21. Highest weight representations

Recall the Killing–Cartan classification (Theorem 54): complex semisimple Lie algebras are in correspondence with root systems (and therefore Dynkin diagrams), via the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_\alpha = \{\, X \in \mathfrak{g} \mid \mathrm{ad}_H(X) = \alpha(H)X \ \forall H \in \mathfrak{h} \,\},$$

where $\mathfrak{h}$ is a Cartan subalgebra.

Note that this is not just a structure theorem for $\mathfrak{g}$, but also for its adjoint representation

$$\mathrm{ad}_{\mathfrak{g}} \colon X \longmapsto \mathrm{ad}_X \in \mathfrak{gl}(\mathfrak{g}).$$

Other representations have a similar structure, as we now show.

Definition 55. Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, and let $\pi \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{g}$.

A *weight* for π is a linear functional $\lambda \colon \mathfrak{h} \to \mathbb{C}$, i.e. $\lambda \in \mathfrak{h}^*$ (the dual space), such that there exists $v \in V \setminus \{0\}$ with

$$\pi(H)v = \lambda(H)v \qquad \text{for all } H \in \mathfrak{h}. \tag{5}$$

The *weight space* corresponding to λ is the subspace V_λ of all $v \in V$ satisfying (5); an element $v \in V_\lambda \setminus \{0\}$ is a *weight vector* for λ.

The *multiplicity* of the weight λ is $\dim V_\lambda$.

Example 18. If π is the adjoint representation, the weights of π are $\Phi \sqcup \{0\}$:

- $\alpha \in \Phi$ has multiplicity 1, $V_\alpha = \mathfrak{g}_\alpha$;
- $\lambda = 0$ has multiplicity n, $V_0 = \mathfrak{h}$.

"Roots are weights of the adjoint representation."

Recall the pairing $\langle \beta, \alpha \rangle := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ on $\mathfrak{h}^*$.

Definition 56. Call an element $\lambda \in \mathfrak{h}^*$ *integral* if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$.

Call $\lambda \in \mathfrak{h}^*$ *dominant* if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Phi^+$.

If $\lambda, \mu \in \mathfrak{h}^*$, then λ is *higher* than μ if $\lambda - \mu$ is a nonnegative linear combination of simple roots.

A weight λ of a $\mathfrak{g}$-representation π is a *highest weight* if λ is higher than every other weight of π, and π is called a *highest-weight representation.*

Theorem 57 (Theorem of the highest weight)**.** *Let $\mathfrak{g}$ be a (finite-dimensional complex) semisimple Lie algebra.*

a) Every $\mathfrak{g}$-irrep π is highest-weight.

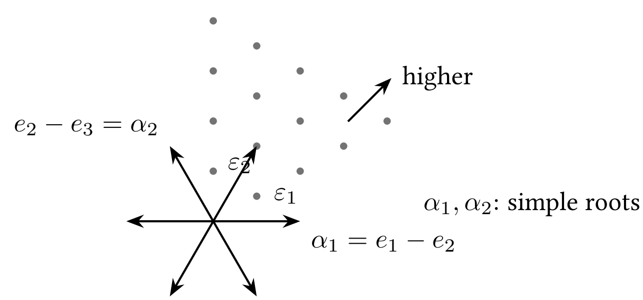
b) π is determined by its highest weight.

c) The highest weights of $\mathfrak{g}$-irreps are precisely the dominant integral elements.

We already proved this for $\mathfrak{g} = \mathfrak{sl}_2$: there, the dominant integral elements correspond to the nonnegative integers.

For concreteness, we will restrict to $\mathfrak{g} = \mathfrak{sl}_3$.

Dominant integral weights:



They are the nonnegative integer combinations of the *fundamental weights*

$$\varepsilon_1 = e_1, \qquad \varepsilon_2 = e_1 + e_2,$$

which satisfy

$$\langle \varepsilon_i, \alpha_j \rangle = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 8. Notice that the highest root (the highest weight of the adjoint representation) is the sum of the fundamental weights.

Here $\mathfrak{h}$ is spanned by

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

A weight is a simultaneous h_1, h_2-eigenvector:

$$\begin{aligned} \varepsilon_1 :\ & h_1\text{-eigenvalue } 1, \quad h_2\text{-eigenvalue } 0, \\ \varepsilon_2 :\ & h_1\text{-eigenvalue } 0, \quad h_2\text{-eigenvalue } 1. \end{aligned}$$

Write $(a, b) := a\varepsilon_1 + b\varepsilon_2$.

Outline of the proof of Theorem 57 [Hal15, §5.4].

Part 1 Every $\mathfrak{g}$-irrep (π, V) has at least one weight.

Part 2 Let λ be a weight for π and $\alpha \in \Phi$. Then, if $v \in V_\lambda$, $e_\alpha v \in V_{\lambda+\alpha}$ and $f_\alpha v \in V_{\lambda-\alpha}$.

Part 3 V is the direct sum of its weight spaces.

Part 4 Call a $\mathfrak{g}$-representation (π, V) *highest-weight cyclic* with weight μ if there exists $v \in V_\mu \setminus \{0\}$ such that $\pi(\mathfrak{g})v = V$ and $\pi(e_1)v = \pi(e_2)v = 0$. In such a representation, μ is the highest weight of π and $\dim V_\mu = 1$.

Part 5 Every $\mathfrak{g}$-irrep is highest-weight cyclic, with a unique highest weight.

Part 6 Every highest-weight cyclic $\mathfrak{g}$-representation is irreducible.

Part 7 Two $\mathfrak{g}$-irreps with the same highest weight are equivalent.

Part 8 All highest weights are dominant integral.

Part 9 All dominant integral elements are highest weights.

22. Properties of highest weight representations

Let (π^λ, V^λ) denote the $\mathfrak{g}$-irrep with highest weight λ.

- The highest weight has multiplicity 1.
- The action of the Weyl group sends weight spaces to weight spaces of the same multiplicity.
- $\mu \in \mathfrak{h}^*$ is a weight for V^λ if and only if μ is contained in the convex hull of $W\lambda$ and $\lambda - \mu$ is an integer linear combination of the simple roots.
- *Weyl character formula*: let

$$\mathrm{ch}_\lambda(H) := \mathrm{tr}\big(e^{\pi_\lambda(H)}\big) = \sum_\mu m_\mu\, e^{\mu(H)} \in \mathbb{C}, \qquad H \in \mathfrak{h},$$

where m_μ is the multiplicity of the weight μ. Then

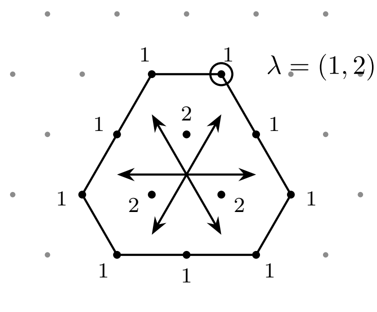
$$\mathrm{ch}_\lambda(H) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)(H)}}{\underbrace{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)(H)}}_{\prod_{\alpha\in\Phi^+}\left(e^{\alpha(H)/2}-e^{-\alpha(H)/2}\right)}}, \qquad \text{where } \rho = \tfrac{1}{2}\sum_{\alpha\in\Phi^+}\alpha.$$

- Can extend to $\mathfrak{g}$ by diagonalizability/continuity.
- With exponentiation, $e^\mu \cdot e^{\mu'} = e^{\mu+\mu'}$ and $\mathrm{ch}_\lambda \cdot \mathrm{ch}_{\lambda'} = \mathrm{ch}_{\lambda+\lambda'} + \cdots$.

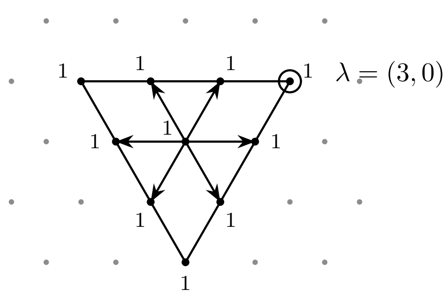
- *Weyl dimension formula*:

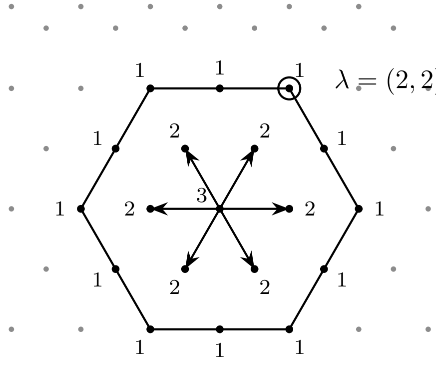
$$\dim V^\lambda = \prod_{\alpha\in\Phi^+}\frac{\langle\lambda+\rho,\alpha\rangle}{\langle\rho,\alpha\rangle}.$$

Examples for $\mathfrak{g} = \mathfrak{sl}_3$:



Exercise 11. Compute $\dim V^\lambda$ for $\lambda = (1,2)$, both from the weight diagram above and from the Weyl dimension formula.





23. Lie groups

The standard references for this section are [Hal15, Ch. 1] and [Bum04, Ch. 5].

Definition 58. A *Lie group* G is a group that is also a differentiable manifold such that the multiplication and inverse maps

$$\begin{aligned} G \times G &\longrightarrow G \\ (g,h) &\longmapsto gh \end{aligned} \qquad \text{and} \qquad \begin{aligned} G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned}$$

are smooth. If G is a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$, it is a *matrix Lie group*.

Examples of matrix Lie groups: $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{C})$, and

- *Orthogonal group*: $\mathrm{O}(n) = \{ g \in \mathrm{GL}_n(\mathbb{C}) \mid gg^T = I \}$.
- *Special orthogonal group*: $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}_n(\mathbb{C})$.
- *Unitary group*: $\mathrm{U}(n) = \{ g \in \mathrm{GL}_n(\mathbb{C}) \mid g\bar{g}^T = I \}$.
- *Special unitary group*: $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}_n(\mathbb{C})$.
- *Symplectic group*: $\mathrm{Sp}_{2n}(\mathbb{F}) = \{ g \in \mathrm{GL}_{2n}(\mathbb{F}) \mid gJg^T = J \}$, where $J = \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$.
- *Compact symplectic group*: $\mathrm{Sp}(2n) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap \mathrm{U}(2n)$.

Lie group	Compact?	Connected?	Simply connected?
$\mathrm{GL}_n(\mathbb{R})$	$\times$	$\times$	n/a
$\mathrm{GL}_n(\mathbb{C})$	$\times$	$\checkmark$	$\times$
$\mathrm{SL}_n(\mathbb{R})$	$\times$	$\checkmark$	$\times$
$\mathrm{SL}_n(\mathbb{C})$	$\times$	$\checkmark$	$\checkmark$
$\mathrm{O}(n)$	$\checkmark$	$\times$	n/a
$\mathrm{SO}(n)$	$\checkmark$	$\checkmark$	$\times$
$\mathrm{U}(n)$	$\checkmark$	$\checkmark$	$\times$
$\mathrm{SU}(n)$	$\checkmark$	$\checkmark$	$\checkmark$
$\mathrm{Sp}_{2n}(\mathbb{R})$	$\times$	$\checkmark$	$\times$
$\mathrm{Sp}_{2n}(\mathbb{C})$	$\times$	$\checkmark$	$\checkmark$
$\mathrm{Sp}(2n)$	$\checkmark$	$\checkmark$	$\checkmark$

24. The matrix exponential and the Lie group–Lie algebra correspondence

Recall (Definition 58) that a Lie group is a group that is also a differentiable manifold such that the multiplication and inverse maps are smooth, and that a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ is a matrix Lie group. Examples of matrix Lie groups: $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{Sp}(2n)$.

The *matrix exponential* is the map

$$\begin{aligned} \mathrm{Mat}_n(\mathbb{C}) &\longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ X &\longmapsto e^X := I + X + \tfrac{1}{2!}X^2 + \tfrac{1}{3!}X^3 + \cdots \end{aligned} \qquad \text{(converges for all } X\text{)}$$

Definition 59 (Def./Prop.)**.** Let G be a matrix Lie group. The *Lie algebra* of G is the set $\mathfrak{g} = \mathrm{Lie}(G)$ of all matrices X such that

$$e^{tX} \in G \qquad \text{for all } t \in \mathbb{R}. \qquad \text{(not } \mathbb{C}\text{!)}$$

$\mathrm{Lie}(G)$ is a Lie algebra in the sense of Definition 49 (for complex Lie groups; for real Lie groups, $\mathrm{Lie}(G)$ is a *real Lie algebra*).

Remark 9.

a) $\{ e^{tX} \mid t \in \mathbb{R} \}$ is called a *one-parameter subgroup*.

b) Since $\{ e^{tX} \mid t \in \mathbb{R} \}$ is connected and $e^{0X} = I$, e^X is an element of the identity component of G.

c) $X = \frac{d}{dt} e^{tX}\big|_{t=0}$, so $\mathrm{Lie}(G)$ is the "tangent space at the identity" of G.

Lie group	Lie algebra
$\mathrm{GL}_n(\mathbb{C})$	$\mathfrak{gl}_n(\mathbb{C})$
$\mathrm{SL}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$
$\mathrm{U}(n)$	$\mathfrak{u}(n) := \left\{ X \in \mathfrak{gl}_n(\mathbb{C}) \;\middle|\; \overline{X}^T + X = 0 \right\}$ (not a complex vector space)
$\mathrm{SU}(n)$	$\mathfrak{su}(n) := \mathfrak{u}(n) \cap \mathfrak{sl}_n(\mathbb{C})$
$\mathrm{O}(n)$	$\mathfrak{so}_n(\mathbb{C})$
$\mathrm{SO}(n)$	$\mathfrak{so}_n(\mathbb{C})$
$\mathrm{Sp}_n(\mathbb{C})$	$\mathfrak{sp}_n(\mathbb{C})$
$\mathrm{Sp}(n)$	$\mathfrak{sp}_n(\mathbb{C}) \cap \mathfrak{u}(n)$

Exercise 12. Work out this relationship for $\mathrm{SL}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$.

Definition 60 (Def./Thm.)**.** Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. A *Lie group homomorphism* is a smooth (holomorphic, if over $\mathbb{C}$) group homomorphism

$$\phi\colon G \longrightarrow H.$$

If $H = \mathrm{GL}_n(\mathbb{C})$, then ϕ is a (complex) *Lie group representation*.

ϕ induces a Lie algebra homomorphism $\phi_*\colon \mathfrak{g} \to \mathfrak{h}$ given by

$$\phi_*(X) = \frac{d}{dt}\phi(e^{tX})\Big|_{t=0},$$

and conversely, $\phi(e^X) = e^{\phi_*(X)}$.

Furthermore, if G is simply connected, then for every Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ there is a unique Lie group homomorphism ϕ such that $\phi(e^X) = e^{\varphi(X)}$.

Example 19. Let

$$\begin{aligned} \mathrm{Ad}\colon G &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\ A &\longmapsto \mathrm{Ad}_A \end{aligned} \qquad \text{where } \mathrm{Ad}_A(X) = AXA^{-1}, \quad X \in \mathfrak{g}.$$

Then Ad_* is the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, since $e^{\mathrm{ad}\, X} = \mathrm{Ad}_{e^X}$ (Problem 24).

Proposition 61. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\Pi, \Pi' \colon G \to \mathrm{GL}(V)$ be Lie group representations and let $\pi, \pi' \colon \mathfrak{g} \to \mathfrak{gl}(V)$ be the corresponding Lie algebra representations. Then*

a) $\Pi(e^X) = e^{\pi(X)}$;

b) $\pi(X) = \frac{d}{dt}\Pi\big(e^{tX}\big)\big|_{t=0}$;

c) Π *is irreducible* $\iff$ π *is irreducible;*

d) $\Pi \cong \Pi'$ $\iff$ $\pi \cong \pi'$;

e) If $X \in \mathfrak{g}$, $A \in G$, *then* $AXA^{-1} \in \mathfrak{g}$ *and*

$$\pi\big(AXA^{-1}\big) = \Pi(A)\,\pi(X)\,\Pi(A)^{-1}.$$

Therefore, if G is simply connected, the irreps of G and $\mathfrak{g}$ are in correspondence.

Let $G = \mathrm{SL}_n(\mathbb{C})$, which is simply connected, with $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \mathrm{Lie}(G)$. For $\alpha \in \Phi$, we have $e_\alpha, f_\alpha, h_\alpha \in \mathfrak{sl}_2 \leq \mathfrak{sl}_3$. Then $\mathrm{SL}_3(\mathbb{C})$ is generated by

$$E_\alpha(t) := \exp(te_\alpha), \qquad F_\alpha(t) := \exp(tf_\alpha), \qquad H_\alpha(t) := \exp(th_\alpha):$$

$$E_{\alpha_1}(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F_{\alpha_1}(t) = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad H_{\alpha_1}(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{\alpha_2}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \quad F_{\alpha_2}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix} \quad H_{\alpha_2}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

$$E_{\rho}(t) = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F_{\rho}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t & 0 & 1 \end{bmatrix} \quad H_{\rho}(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Let V^λ be the highest weight representation associated to λ, and let μ be a weight with weight vector v_μ. Then

$$\begin{aligned} \Pi\big(e^{th}\big)v_\mu = e^{\pi(th)}v_\mu &= \sum_{m\geq 0} \frac{\pi(th)^m}{m!}v_\mu \\ &= \sum_{m\geq 0} \frac{\mu(th)^m}{m!}v_\mu = \sum_{m\geq 0} \frac{t^m\mu(h)^m}{m!}v_\mu \\ &= e^{t\mu(h)}v_\mu \qquad \text{(still an eigenvector)} \end{aligned}$$

However,

$$\begin{aligned} \Pi\big(e^{te_\alpha}\big)v_\mu = e^{\pi(te_\alpha)}v_\mu &= \sum_{m\geq 0} \frac{\pi(te_\alpha)^m}{m!}v_\mu \\ &= \sum_{m\geq 0} \frac{t^m}{m!}\underbrace{\pi(e_\alpha)^m v_\mu}_{\in V_{\mu+m\alpha}} \qquad \text{(not cleanly a raising operator)} \end{aligned}$$

25. Compact groups and the Peter–Weyl theorem

Definition 62. A *topological group* is a group that is also a (Hausdorff) topological space such that the multiplication and inverse maps are continuous. All maps (e.g. representations) are assumed to be continuous.

Some important cases:

$$\begin{array}{ccc} \text{Compact Lie groups} & \subseteq & \text{Lie groups} \\ \cup\!\mid & & \cup\!\mid \\ \text{Compact groups} & \subseteq & \underbrace{\text{locally compact groups}}_{\substack{\text{small-enough closed} \\ \text{neighbourhoods are compact}}} \end{array}$$

Definition 63 (Def./Thm.)**.**

a) A *Borel measure* is a measure that is defined on all open sets (and thus all *Borel sets*). Any locally compact Hausdorff space has a Borel measure.

b) A *left Haar measure* on a locally compact group G is a Borel measure μ_L that is invariant under left translation:

$$\mu_L(X) = \mu_L(gX) \qquad \text{for all measurable } X \text{, and } g \in G.$$

A *right Haar measure* on G is a Borel measure μ_R that is invariant under right translation:

$$\mu_R(X) = \mu_R(Xg) \qquad \text{for all measurable } X \text{, and } g \in G.$$

Furthermore:

c) Every locally compact group has a left/right Haar measure, unique up to scalar multiple.

d) For compact groups, $\mu_L = \mu_R =: \mu$.

Proof. c) See sources in [Bum04, Ch. 1]. d) [Bum04, Prop. 1.1, 1.2]. □

We use this measure to integrate measurable functions on G:

$$\int_G f(g)\, dg := \int_G f(g)\, d\mu(g), \qquad (\mu = \mu_L \text{ or } \mu_R)$$

When G is compact, let us normalize μ so that $\mu(G) = 1$.

Recall

$$L^2(G) = \left\{ f\colon G \to \mathbb{C} \;\middle|\; \int_G |f(g)|^2\, dg < \infty \right\},$$

with Hermitian inner product $\langle f, f' \rangle := \int_G f(g)\overline{f'(g)}\, dg$ and G-action $g \cdot f(g') := f(g^{-1}g')$. This is the most natural analogue of the regular representation (Problem 4).

If G is a compact abelian group, then all finite-dimensional irreps of G are 1-dimensional (see Problem 21) and any $f \in L^2(G)$ has a "Fourier expansion"

$$f(g) = \sum_{\chi\colon \text{irred}} a_\chi \chi(g), \qquad a_\chi = \int_G \underbrace{f(g)\overline{\chi(g)}}_{L^2\text{-inner product}}\, dg,$$

and

$$\int_G |f(g)|^2 \, dg = \sum_\chi |a_\chi|^2 \qquad \text{(Plancherel formula)}$$

We want to show something similar for general compact groups.

Theorem 64. *Let G be a compact group. Then every finite-dimensional representation V is the direct sum of irreps.*

Proof. Use Weyl's averaging trick. Let $\langle \cdot, \cdot \rangle$ denote any Hermitian inner product on V. Define

$$\langle v, w \rangle_G := \int_G \langle gv, gw \rangle \, dg.$$

Now, $\langle \cdot, \cdot \rangle_G$ is also a Hermitian inner product: conjugate symmetric, linear in the first argument, positive definite. It is also G-invariant:

$$\begin{aligned} \langle gv, gw \rangle_G &= \int_G \langle hgv, hgw \rangle \, d\mu(h) \\ &= \int_G \langle h'v, h'w \rangle \, d\mu(h') \qquad (h' = hg; d\mu(h) = d\mu(gh)) \\ &= \langle v, w \rangle_G \,. \end{aligned}$$

If $W \leq V$ is a subrepresentation, then $U := W^\perp$ (with respect to $\langle \cdot, \cdot \rangle_G$) is also a subrepresentation, since if $u \in U$, $w \in W$, then

$$\langle gu, w \rangle_G = \langle u, \underbrace{g^{-1}w}_{\in W} \rangle_G = 0.$$

Thus every subrepresentation has an orthogonal complement, so V is completely reducible. □

(Compare to the proof of Maschke's Theorem, Theorem 5.) We have Schur's Lemma and character orthogonality similarly to the finite case.

Definition 65. A *matrix coefficient* for a representation (Π, V) of a group G is a function of the form

$$\phi \colon G \longrightarrow \mathbb{C}, \qquad \phi(g) := L\big(\Pi(g)v\big),$$

where L is a linear functional $V \to \mathbb{C}$.

Example 20. If $v = e_i$, the ith basis vector, and $L = e_j^*$ is the functional sending $v \in V$ to its jth component, then $\phi(g)$ is the (j, i)-entry of $\Pi(g)$.

Definition 66. A representation of a group G on a complex Hilbert space V (a complete finite- or infinite-dimensional Hermitian inner product space) is *unitary* if $\langle gv, gw \rangle = \langle v, w \rangle$ for all $g \in G$, $v, w \in V$.

We have seen that finite-dimensional representations of finite and compact groups are *unitarizable* (able to be made unitary).

Theorem 67 (Peter–Weyl Theorem)**.** *Let G be a compact group.*

a) *The matrix coefficients of G for finite-dimensional representations are dense in the set*

$$C(G) = \{\, f\colon G \to \mathbb{C} \mid f \text{ continuous} \,\}$$

of continuous functions on G, with respect to the L^∞-norm

$$\left|f - f'\right|_\infty = \sup_{g \in G} \left|f(g) - f'(g)\right|$$

(and therefore in $L^2(G)$, since $C(G)$ is dense in $L^2(G)$).

b) *Any unitary representation of a compact group on a complex Hilbert space is a direct sum of finite-dimensional irreducible representations.*

c) *We have the (completed) direct sum decomposition*

$$L^2(G) = \widehat{\bigoplus_\pi} V_\pi^{\oplus \dim V_\pi},$$

where the sum is over all unitary irreps (π, V_π).

We will use convolution of functions:

$$(f * f')(g) := \int_G f(gh^{-1}) f'(h)\, dh = \int_G f(h) f'(h^{-1}g)\, dh,$$

which is always defined if $f, f' \in C(G)$.

If $\phi \in C(G)$, let T_ϕ be the $L^2(G)$-operator

$$(T_\phi f) = \phi * f,$$

and let

$$V(\lambda) := V(\lambda, \phi) = \left\{\, f \in L^2(G) \;\middle|\; T_\phi f = \lambda f \,\right\}$$

be the λ-eigenspace.

T_ϕ is bounded, satisfying $|T_\phi f|_\infty \le |\phi|_\infty |f|_1$ for all $f \in L^1(G)$, and self-adjoint if $\phi(g^{-1}) = \overline{\phi(g)}$. Using the Spectral Theorem for compact operators, the $V(\lambda)$ are orthogonal, span $L^2(G)$, and if $\lambda \neq 0$, $V(\lambda)$ is finite-dimensional; in fact, if $q > 0$, $\sum_{|\lambda| > q} \dim V(\lambda) < \infty$. ([Bum04, Ch. 3–4])

In addition it is invariant under right multiplication ($\rho(g) \cdot f(h) = f(hg)$): if $T_\phi f = \lambda f$, then

$$\big(T_\phi \rho(g) f\big)(x) = \int_G \phi(xh^{-1}) f(hg)\, dh = \int_G \phi(xgh^{-1}) f(h)\, dh = \rho(g)(T_\phi f)(x) = \big(\lambda \rho(g) f\big)(x).$$

Proof sketch of Theorem 67 a). Let $f \in C(G)$ and $\varepsilon > 0$. We will show that there exists a matrix coefficient f' with $|f - f'|_\infty < \varepsilon$.

Since f is continuous on a compact set, there is a neighbourhood U of $1 \in G$ such that

$$|g \cdot f - f|_\infty < \varepsilon/2 \qquad \text{for all } g \in U.$$

Let ϕ be a nonnegative function supported on U such that $\int_G \phi(g)\,dg = 1$ and $\phi(g^{-1}) = \overline{\phi(g)}$. Then if $h \in G$,

$$\begin{aligned} |T_\phi f(h) - f(h)| &= \left| \int_G \big(\phi(g) f(g^{-1}h) - \phi(g) f(h)\big)\,dg \right| \\ &\le \int_U \phi(g) \left| f(g^{-1}h) - f(h) \right| \,dg \\ &\le \int_U \phi(g) \left| g \cdot f - f \right|_\infty \,dg \\ &\le \int_U \phi(g) \cdot \tfrac{\varepsilon}{2}\,dg = \tfrac{\varepsilon}{2}, \end{aligned}$$

so $|T_\phi f - f|_\infty < \varepsilon/2$.

Let f_λ be the orthogonal projection onto $V(\lambda) \le L^2(G)$. Then the f_λ are orthogonal, so

$$\sum_\lambda |f_\lambda|_2^2 = |f|_2^2 < \infty, \qquad \text{where } |f|_2 = \left(\int_G |f(x)|^2 \,dx \right)^{1/2}. \tag{6}$$

For $q > 0$, let

$$f'' = \sum_{|\lambda| > q} f_\lambda, \qquad f' = T_\phi(f'').$$

We have $f', f'' \in \bigoplus_{|\lambda|>q} V(\lambda) =: V_q$, which is finite-dimensional.

We claim that every $F \in V_q$ is a matrix coefficient. Since $V(\lambda)$ is right-multiplication invariant, $\rho(g)F \in V_q$ for all $g \in G$. Then $W = \operatorname{span}\{\, \rho(g)F \mid g \in G \,\} \le V_q$ is a finite-dimensional G-representation. Let $L\colon W \to \mathbb{C}$, $L(\phi) := \phi(1)$. Then

$$L\big(\rho(g)F\big) = \big(\rho(g)F\big)(1) = F(g),$$

so F is a matrix coefficient, proving the claim.

By (6) we can choose q such that

$$\Big| \sum_{0<|\lambda|<q} f_\lambda \Big|_1 \le \Big| \sum_{0<|\lambda|<q} f_\lambda \Big|_2 < \frac{\varepsilon}{2\,|\phi|_\infty}.$$

We have

$$T_\phi(f - f'') = T_\phi\Big(\sum_{0 \le |\lambda| < q} f_\lambda \Big) = T_\phi\Big(\sum_{0<|\lambda|<q} f_\lambda \Big),$$

and so

$$\big|T_\phi(f - f'')\big|_\infty \le |\phi|_\infty \Big| \sum_{0<|\lambda|<q} f_\lambda \Big|_1 < \frac{\varepsilon}{2}.$$

Hence

$$\big|f - f'\big|_\infty \le |f - T_\phi f|_\infty + \big|T_\phi f - T_\phi f''\big|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Proof of Theorem 67 b). Let G be a compact group and $V \ne 0$ be a unitary representation. Let $v \in V \setminus \{0\}$. Let N be a neighbourhood of 1 in G such that $|\Pi(g)v - v| \le |v|/2$ for all $g \in N$. Let $\phi \in C(G)$ have $\operatorname{supp}\phi \subseteq N$ and $\int_G \phi(g)\,dg = 1$.

Now,

$$\left\langle \int_G \phi(g)\Pi(g)v\,dg,\ v\right\rangle = \langle v, v\rangle - \left\langle \int_N \phi(g)\big(v - \Pi(g)v\big)\,dg,\ v\right\rangle,$$

and

$$\left|\left\langle \int_N \phi(g)\big(v - \Pi(g)v\big)\,dg,\ v\right\rangle\right| \le \int_N |v - \Pi(g)v|\,dg\cdot|v| \le \frac{|v|^2}{2},$$

so $\int_G \phi(g)\Pi(g)v\,dg \neq 0$.

Let $\varepsilon > 0$ and, by a), choose a matrix coefficient f such that $|f-\phi|_\infty < \varepsilon$. By unitarity,

$$\left|\int_G (f-\phi)(g)\Pi(g)v\,dg\right| \le \varepsilon\,|v|\,,$$

so for small enough ε, $\int_G f(g)\Pi(g)v\,dg \neq 0$.

Since f is a matrix coefficient, so is $g \mapsto f(g^{-1})$ [Bum04, Prop. 2.4]. So write $f(g^{-1}) = L\big(\rho(g)w\big)$, where (ρ, W) is a finite-dimensional representation, $w \in W$, and $L\colon W \to \mathbb{C}$ is a linear functional. Define the map $T\colon W \to V$ by

$$T(x) = \int_G L\big(\rho(g^{-1})x\big)\Pi(g)v\,dg.$$

T is G-equivariant by the usual change-of-variables argument, and $T \neq 0$ since

$$T(w) = \int_G f(g)\Pi(g)v\,dg \neq 0.$$

Thus $\operatorname{im} T$ is a nonzero finite-dimensional subrepresentation, so V has a finite-dimensional irreducible subrepresentation. Then by Zorn's Lemma, the maximal direct sum of orthogonal finite-dimensional irreps is V itself. □

Proof of Theorem 67 c). The argument is the one used for finite groups in Problem 26. □

26. Complete reducibility, and polynomial representations of $\mathrm{GL}_n(\mathbb{C})$

Section 25 developed the representation theory of compact groups; in particular, their finite-dimensional representations are completely reducible.

Definition 68. Let G be a complex reductive group.

a) G is *semisimple* if its Lie algebra is semisimple.

b) G is *reductive* if its Lie algebra is reductive.

(There are several variants of these definitions.)

All of the complex Lie groups of Section 23 are reductive.

Theorem 69 (Weyl's Theorem)**.** *All finite-dimensional representations of complex connected semisimple Lie algebras and Lie groups are completely reducible.*

We will say that such an object has *complete reducibility.*

Very rough proof sketch. See [Hal15, §6.1].

a) Compact groups have complete reducibility (Theorem 64).
b) So Lie algebras of simply connected compact Lie groups have complete reducibility, using the Lie group–Lie algebra correspondence (Proposition 61).
c) Every complex semisimple Lie algebra is the "complexification" of a Lie algebra from part b). Furthermore, complexification preserves complete reducibility.
d) So again using the Lie group–Lie algebra correspondence (Proposition 61), all complex connected semisimple Lie groups have complete reducibility. □

What about the reductive case?

Example 21. $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{z}$, where

$$\mathfrak{z} = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \,\middle|\, a \in \mathbb{C} \right\}.$$

The map

$$\pi \colon \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \longmapsto \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$$

is a Lie algebra representation, but

$$\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ay \\ 0 \end{bmatrix},$$

so the only eigenvector is $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. So neither $\mathfrak{z}$ nor $\mathfrak{gl}_n(\mathbb{C})$ have complete reducibility!

Let us try to exponentiate this representation using Proposition 61:

$$e^{\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]} = \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix},$$

$$\Pi\left(\begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix} \right) = e^{\pi\left(\left[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right]\right)} = e^{\left[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right]} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix},$$

i.e.

$$\Pi\left(\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \right) = \begin{bmatrix} 1 & \log b \\ 0 & 1 \end{bmatrix} \qquad \text{(not a (continuous) representation!)}$$

Note that $\mathrm{GL}_n(\mathbb{C})$ (and $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$) are not simply connected.

It turns out that $\mathrm{GL}_n(\mathbb{C})$ *does* have complete reducibility:

$$\mathrm{GL}_n(\mathbb{C}) = \big(\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})\big)/\mu_n, \qquad (n\text{th roots of } 1)$$

Thus every $\mathrm{GL}_n(\mathbb{C})$-representation can be inflated to a $\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$-representation via

$$\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) \twoheadrightarrow \mathrm{GL}_n(\mathbb{C}) \to V.$$

The (holomorphic) irreps of $\mathbb{C}^\times$ are $z \mapsto z^n$, and the irreps of $\mathrm{SL}_n(\mathbb{C})$ are the highest weight representations (exponentiated from the highest weight representations of $\mathfrak{sl}_n(\mathbb{C})$).

So the $\mathrm{GL}_n(\mathbb{C})$-irreps turn out to be of the form

$$\underbrace{\det{}^{\otimes N}}_{\substack{\mathbb{C}^\times\text{-representation} \\ \text{“central character”}}} \otimes \underbrace{V^\lambda}_{\substack{\text{highest} \\ \text{weight representation}}}$$

$(\det\colon g \mapsto \det(g))$

(λ a dominant weight)

and all (finite-dimensional, holomorphic) $\mathrm{GL}_n(\mathbb{C})$-representations are direct sums of these representations.

Now, dominant weights in type A_{n-1} are of the form

$$\begin{aligned} \lambda &= a_1 e_1 + a_2(e_1 + e_2) + \cdots + a_{n-1}(e_1 + \cdots + e_{n-1}) \\ &= (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}), \qquad \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0, \quad \lambda_i \in \mathbb{Z}. \end{aligned}$$

Integer partitions with $\leq n-1$ parts!

Remember that we view e_i as the linear functional

$$e_i \begin{bmatrix} \ddots & & \\ & a_i & \\ & & \ddots \end{bmatrix} = a_i.$$

In $\mathfrak{sl}_n(\mathbb{C})$, $e_n = -e_1 - \cdots - e_{n-1}$. But in $\mathfrak{gl}_n(\mathbb{C})$ it is linearly independent, and we have the weight

$$(\lambda_1, \lambda_2, \ldots, \lambda_n), \qquad \lambda_i \in \mathbb{Z}.$$

Central character:

$$\lambda\left(\begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix}\right) = a(\lambda_1 + \cdots + \lambda_n).$$

Tensoring by $\det$ corresponds to adding 1 to each λ_i. So dominant means

$$\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n,$$

and tensoring by $\det^{\otimes(-\lambda_n)}$ gives back the $\mathfrak{sl}$ condition $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0$.

Exponentiating yields

$$\begin{tikzcd} \mathrm{diag}(a_1, \ldots, a_n) \arrow[r, mapsto] \arrow[d, "\exp"'] & a_1\lambda_1 + \cdots + a_n\lambda_n \arrow[d, "\exp"] \\ \mathrm{diag}(b_1, \ldots, b_n) \arrow[r, mapsto] & b_1^{\lambda_1} \cdots b_n^{\lambda_n} \end{tikzcd}$$

and further arguments show that all the matrix coefficients are Laurent polynomials in the matrix entries.

Definition 70 (Def./Thm.). A representation π of $\mathrm{GL}_n(\mathbb{C})$ is *polynomial* if all its matrix coefficients $g \mapsto \langle e_j, \pi(g)e_i\rangle$ are polynomial functions (in the entries) of g. π is *rational* if its matrix coefficients are rational functions of g.

If $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n$, is a dominant weight, then the (exponentiated) highest weight representation V^λ is rational, and if $\lambda_n \geq 0$, it is polynomial. Every rational representation can be written in the form $\det^{\otimes(-a)} \otimes V^\lambda$ where V^λ is polynomial.

27. The double commutant theorem and Schur–Weyl duality

Let V be a finite-dimensional vector space. If $A \subseteq \operatorname{End} V$, let

$$\operatorname{Comm}(A) = \{ b \in \operatorname{End} V \mid ab = ba \; \forall a \in A \} .$$

Theorem 71 (Double commutant/centralizer theorem)**.** *Let $A \subseteq \operatorname{End}(V)$ be a subalgebra such that V is a completely reducible A-module. Let $B = \operatorname{Comm}(A)$.*

a) $\operatorname{Comm}(B) = A$*.*

b) *V is completely reducible as a B-module.*

c) *As an $(A \otimes B)$-module, we have the decomposition*

$$V \cong \bigoplus_i U_i \otimes W_i,$$

where U_i (resp. W_i) are the distinct A-irreps (resp. B-irreps) appearing in V. The U_i and W_i determine each other, and

$$\begin{aligned} \operatorname{Hom}(U_i, V) &\cong W_i \quad \text{as } B\text{-modules}, \\ \operatorname{Hom}(W_i, V) &\cong U_i \quad \text{as } A\text{-modules}. \end{aligned}$$

Proof. See [GW09, Thms. 4.1.13, 4.2.1]. ☐

Set $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ and let $V = \mathbb{C}^n$ be the standard representation. Then

$$V^{\otimes k} = \operatorname{span} \{ v_1 \otimes \cdots \otimes v_k \mid v_i \in V \}$$

is a GL_n-representation ρ under

$$g \cdot (v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k$$

and an S_k-representation π under

$$w \cdot (v_1 \otimes \cdots \otimes v_k) = v_{w^{-1}(1)} \otimes \cdots \otimes v_{w^{-1}(k)}.$$

Let

$$A = \operatorname{span}\big(\rho(\mathrm{GL}_n)\big) \subseteq \operatorname{End}\big(V^{\otimes k}\big), \qquad B = \operatorname{span}\big(\pi(S_k)\big) \subseteq \operatorname{End}\big(V^{\otimes k}\big).$$

Proposition 72. *We have $\operatorname{Comm}(A) = B$ and $\operatorname{Comm}(B) = A$. As a $(\mathrm{GL}_n \times S_k)$-representation,*

$$V^{\otimes k} \cong \bigoplus_i E^i \otimes F^i, \tag{7}$$

where the E^i are distinct GL_n-irreps and the F^i are distinct S_k-irreps. (Later, we will see which ones.)

Proof. Since V is finite-dimensional and since GL_n and S_k have complete reducibility, Theorem 71 applies, and if we can show that $\operatorname{Comm}(A) = B$ the rest of the statements follow.

By inspection, $B \subseteq \operatorname{Comm}(A)$ and $A \subseteq \operatorname{Comm}(B)$. Thus it suffices to show that $\operatorname{Comm}(B) \subseteq A$.

Let $\{e_1, \dots, e_n\}$ be the standard basis of V. For $I = (i_1, \dots, i_k)$, let $|I| = k$ and $e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}$. Then $\{ e_I \mid |I| = k \}$ is a basis for $V^{\otimes k}$, and the S_k-action can be written

$$we_I = e_{wI}, \qquad \text{where } w \cdot I = (i_{w^{-1}(1)}, \dots, i_{w^{-1}(k)}).$$

Write $T \in \operatorname{End}(V^{\otimes k})$ as a matrix relative to this basis:

$$Te_J = \sum_I a_{I,J} e_I.$$

Then

$$T(w \cdot e_J) = T(e_{wJ}) = \sum_I a_{I,wJ} e_I$$

and

$$w \cdot T(e_J) = \sum_I a_{I,J} e_{wI} = \sum_I a_{w^{-1}I,J} e_I,$$

so

$$\begin{aligned} T \in \operatorname{Comm}(B) &\iff a_{I,wJ} = a_{w^{-1}I,J} \quad \forall w, I, J \\ &\iff a_{wI,wJ} = a_{I,J} \quad \forall w, I, J. \end{aligned} \tag{8}$$

Consider the bilinear form $(X, Y) := \operatorname{tr}(XY)$ on $\operatorname{End}(V^{\otimes k})$. This is nondegenerate, i.e. $(X, Y) = 0$ for all X implies $Y = 0$.

We can project onto $\operatorname{Comm}(B)$ by averaging:

$$X \longmapsto X^\natural = \frac{1}{k!} \sum_{w \in S_k} \pi(w) X \pi(w)^{-1} \in \operatorname{Comm}(B),$$

and if $T \in \operatorname{Comm}(B)$, then

$$(X^\natural, T) = \frac{1}{k!} \sum_{w \in S_k} \operatorname{tr}\big(\pi(w) X \pi(w)^{-1} T\big) = (X, T),$$

since the trace is cyclically invariant and T commutes with π. Thus if $(X, T) = 0$ for all $X \in \operatorname{Comm}(B)$, then $(X, T) = 0$ for all $X \in \operatorname{End}(V^{\otimes k})$, so since $(\cdot, \cdot)$ is nondegenerate, $T = 0$. This means $(\cdot, \cdot)$ is nondegenerate on $\operatorname{Comm}(B)$.

To show that $\operatorname{Comm}(B) = A$, it thus suffices to show that if $T \in \operatorname{Comm}(B)$ satisfies $(X, T) = 0$ for all $X \in A$ (i.e. for all $X \in \mathrm{GL}_n$), then $T = 0$.

If $g = [g_{ij}] \in \mathrm{GL}_n$, then $\rho(g) = [g_{I,J}]$ where $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_k)$ and $g_{I,J} := g_{i_1,j_1} \cdots g_{i_k,j_k}$. Now, we assume

$$0 = \big(T, \rho(g)\big) = \sum_{I,J} a_{I,J}\, g_{J,I} \qquad \forall g \in \mathrm{GL}_n\,. \tag{9}$$

By continuity, (9) also holds for all $X = [x_{ij}] \in \operatorname{Mat}_n(\mathbb{C})$. By (8), if $X \in \operatorname{Mat}_n(\mathbb{C}) \subseteq \operatorname{Comm}(B)$, then

$$x_{J,I} = x_{wJ,wI} \qquad \forall w \in S_k.$$

So we have

$$0 = \sum_{I,J} a_{I,J} x_{J,I} = \sum_\gamma n_\gamma a_\gamma x_\gamma,$$

where the sum is over a set of representatives $\gamma = (I, J)$ of the S_k-orbits of the set of ordered pairs (I, J); $a_\gamma := a_{I,J}$ and $x_\gamma := x_{J,I}$ are constant on these orbits, and we set $n_\gamma := |S_k \cdot \gamma|$.

Think of this as a polynomial function in the n^2 variables $[x_{ij}]$. Then the x_γ are linearly independent, so $a_\gamma = 0$ for all γ, and therefore $T = 0$. Hence $A = \mathrm{Comm}(B)$. $\square$

It remains to identify the representations E^i and F^i appearing in (7).
Let $T \cong (\mathbb{C}^\times)^n \le \mathrm{GL}_n$ be the subgroup of diagonal matrices. Since T is abelian, by Problem 21 $V^{\otimes k}$ is a direct sum of T-weight spaces:

$$V^{\otimes k} = \bigoplus_{\mu\colon \text{weight}} V_\mu, \qquad \begin{aligned} \mathrm{diag}(z_1, \ldots, z_n)v &= \mu(z_1, \ldots, z_n)v \\ &= z_1^{\mu_1} \cdots z_n^{\mu_n} v \qquad \forall v \in V_\mu. \end{aligned}$$

Here μ is a composition $(\mu_1, \ldots, \mu_n)$, $\mu_1 + \cdots + \mu_n = k$. In fact,

$$V_\mu = \mathrm{span}\,\{\, e_I \mid \mu_I = \mu \,\}, \qquad \text{where } (\mu_I)_j = \#\, e_j \text{ appearing in } e_I.$$

So $V_\mu \neq 0 \iff \mu \in \mathrm{Comp}(k, n)$, the compositions of k with n parts.

In particular (using Definition 70), every E^i is a *polynomial* representation of GL_n.

Since the actions of T and S_k commute, V_μ is an S_k-module for all $\mu \in \mathrm{Comp}(k, n)$. Basis vectors $e_I \in V_\mu$, $I = (i_1, \ldots, i_n)$ with μ_j copies of j, are in (S_k-equivariant) bijection with diagrams

$$\begin{array}{lll} a_1\ a_2\ \cdots & \leftarrow \text{positions of 1's in } I\text{, in increasing order} & \\ b_1\ b_2\ \cdots & \leftarrow \qquad\qquad\qquad\quad " & \text{2's} \\ c_1\ c_2\ \cdots & \leftarrow \qquad\qquad\qquad\quad " & \text{3's} \end{array}$$

Tabloids! Thus, we have proved:

Lemma 73. *As an S_k-module,*

$$V_\mu \cong M^\mu,$$

where M^μ is the span of μ-tabloids with action as in Definition 24.

Note that $M^\mu \cong M^\lambda$ for any rearrangement λ of the parts of μ.

Now, since every S^λ appears in some M^μ and by Theorem 40, we have $|\lambda| = |\mu| = k$, $\ell(\lambda) \le \ell(\mu) = n$, and $\lambda \trianglerighteq \mu$.

Since the V_μ span $V^{\otimes k}$ and since the F^i in (7) are distinct S_k-irreps, we can rewrite (7):

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \mathrm{Par}(k,n)} E^\lambda \otimes S^\lambda,$$

where $\mathrm{Par}(k, n)$ is the set of partitions of k with length $\le n$.

It remains to determine the E^λ. Fix $\lambda \in \mathrm{Par}(k, n)$. Let v be a GL_n-highest-weight vector in $E^\lambda \otimes S^\lambda$, and let μ be its weight, so that $v \in V_\mu \cong M^\mu$. We will show that μ can be taken to equal λ, and by uniqueness, it must be λ.

As in the proof of Theorem 40, M^μ can be identified with $\mathbb{C}[\mathcal{T}_{\lambda\mu}]$, the span of tableaux of shape λ, content μ. Furthermore, $v \in S^\lambda \le M^\mu$ is a linear combination of elements

$$v = \sum_{w,T} c_{w,T}\, v_{w,T}, \qquad v_{w,T} := \kappa_{t_\lambda} \sum_{S \in \{T\}} wS,$$

where t_λ is the tableau of shape λ numbered $1, 2, \ldots$ in reading order. Unless $\lambda \trianglerighteq \mu$, $v_{T,T'} = 0$ for all $T, T' \in \mathbb{C}[\mathcal{T}_{\lambda\mu}]$.

Since v is highest-weight, it is highest weight for the corresponding $\mathfrak{gl}_n$-representation, so the raising operators

$$X_i = E_{i,i+1} = \begin{array}{cc} \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} & \begin{array}{c} \\ i \\ i+1 \end{array} \end{array} \qquad \text{(usually called } e_i\text{, but changed for notation reasons)}$$

have $X_i v = 0$ for all i. Here the action is

$$X_i(S) = \sum S',$$

where $S' \in R_i(S) := \{\text{tableaux formed from } S \text{ by changing an } i+1 \text{ to an } i\}$.

We have $S \in \mathcal{T}_{\lambda\mu} \implies S' \in \mathcal{T}_{\lambda\nu}$, where

$$\nu = (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1} - 1, \mu_{i+2}, \ldots, \mu_n) \ntrianglelefteq \mu.$$

Thus

$$X_i(v_{w,T}) = \kappa_{t_\lambda} \sum_{S \in \{T\}} w\, X_i(S) = v_{w,X_i(T)} \in S^\lambda \cap M^\nu.$$

So taking $\lambda = \mu$ gives $v_{w,X_i(T)}$ for all i, so v is highest weight.

We have proved:

Theorem 74 (Schur–Weyl duality). *As a* $(\mathrm{GL}_n \times S_k)$*-representation,*

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \mathrm{Par}(k,n)} V^\lambda \otimes S^\lambda.$$

If $n \geq k$, then all irreps of S_k appear.

28. Symmetric functions

Corollary 75. *In the highest weight representation V^λ, the weight space multiplicities are given by the Kostka numbers:*

$$\underbrace{\dim V^\lambda_\mu}_{\substack{\text{weight space for } \mu \\ \text{inside } V^\lambda}} = k_{\lambda\mu}.$$

Proof. Using Schur–Weyl duality, treating $V^{\otimes k}$ as an S_k-representation, we have

$$V^{\otimes k} \cong \bigoplus_{\lambda \in \mathrm{Par}(k,n)} V^\lambda \otimes S^\lambda \cong \bigoplus_{\lambda,\mu} \Big(V_\mu \cap \big(V^\lambda \otimes S^\lambda\big)\Big) \cong \bigoplus_{\lambda,\mu} (V^\lambda_\mu \otimes S^\lambda),$$

and on V_μ, using Lemma 73, this gives

$$\bigoplus_\lambda k_{\lambda\mu} S^\lambda \cong M^\mu \cong V_\mu \cong \bigoplus_\lambda (V^\lambda_\mu \otimes S^\lambda) \cong \bigoplus_\lambda (\dim V^\lambda_\mu) S^\lambda. \qquad \square$$

Symmetric functions [Mac95], [Sta99].

Definition 76. The *ring of symmetric functions* is the ring $\Lambda \subseteq \mathbb{C}[[x_1, x_2, \ldots]]$ consisting of power series of bounded degree that are symmetric under permutations of the x_i.

Setting $x_{n+1} = x_{n+2} = \cdots = 0$ gives symmetric polynomials in n variables.

Several important bases, all homogeneous of degree $|\lambda|$:

- *elementary symmetric functions*

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}, \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$$

- *complete homogeneous symmetric functions*

$$h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}, \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$$

- *monomial symmetric functions*

$$m_\lambda = \sum_{\substack{i_1, \ldots, i_\ell \\ \text{distinct}}} x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}$$

- *power sum symmetric functions*

$$p_k = \sum_{i \geq 1} x_i^k, \qquad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

- *Schur functions*

$$s_\lambda = \sum_{\substack{\text{SSYT } T \\ \text{of shape } \lambda}} x^{\mathrm{wt}(T)} = \sum_{\mu\colon \text{ comp.}} k_{\lambda\mu}\, x^\mu = \sum_{\substack{\mu\colon \text{ part.} \\ \mu \trianglelefteq \lambda}} k_{\lambda\mu}\, m_\mu,$$

  with $s_{(k)} = h_k$ and $s_{(1^k)} = e_k$.

Schur functions satisfy several nice identities, including (see e.g. [Mac95, Ch. 1]):

- *Cauchy bialternant / Weyl character*:

$$s_\lambda(x_1, \ldots, x_n) = \frac{\sum_{w \in S_n} (-1)^w x^{w(\lambda+\rho)}}{\sum_{w \in S_n} (-1)^w x^{w\rho}} = \frac{a_{\lambda+\rho}}{a_\rho}, \qquad a_\rho = \prod_{i<j} (x_i - x_j).$$

  In particular, it is the character of the highest weight representation V^λ.

- *Cauchy identity*:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y) = \sum_\mu \frac{1}{z_\mu} p_\mu(x) p_\mu(y),$$

  where $z_\mu = \mu_1 \mu_2 \cdots \underbrace{m_i(\mu)}_{\substack{\text{number of parts} \\ \text{of size } i}}! \cdots$

- *Pieri rule*:
$$h_k s_\mu = \sum_\lambda s_\lambda, \qquad e_k s_{\mu'} = \sum_\lambda s_{\lambda'},$$
where the sums are over all λ such that $|\lambda| = |\mu| + k$ and $\lambda'_i - \mu'_i \le 1$ for all i.

- *Jacobi–Trudi rule*:
$$s_\lambda = \det\left[h_{\lambda_i+j-i}\right]_{1\le i,j\le \ell(\lambda)} = \det\left[e_{\lambda'_i+j-i}\right]_{1\le i,j\le \ell(\lambda')}.$$

Proposition 77. *Let $g \in \mathrm{GL}_n$ have eigenvalues $x_1, \dots, x_n$, and let $w \in S_k$ have cycle type $\mu = (\mu_1, \dots, \mu_\ell)$. Then*
$$\mathrm{tr}_{V^{\otimes k}}(g, w) = p_\mu(x_1, \dots, x_n).$$

Proof. By continuity and conjugation, we can assume $g = \mathrm{diag}(x_1, \dots, x_n)$. Then
$$(g, w) \cdot e_I = x_{i_1} \cdots x_{i_k} e_{w\cdot I}, \qquad \text{where } w \cdot I = (i_{w^{-1}(1)}, \dots, i_{w^{-1}(k)}).$$

Now, $e_{w\cdot I} = e_I$ if and only if i_j is constant on each cycle of w. For such an I, let $j_1, \dots, j_\ell$ denote the values of i on each conjugacy class. Thus we have
$$\begin{aligned}
\mathrm{tr}_{V^{\otimes k}}(g, w) &= \sum_{\substack{I \\ w\cdot I = I}} x_{i_1} \cdots x_{i_k} \\
&= \sum_{1\le j_1,\dots,j_\ell \le n} x_{j_1}^{\mu_1} \cdots x_{j_\ell}^{\mu_\ell} \\
&= p_{\mu_1}(x)\, p_{\mu_2}(x) \cdots p_{\mu_\ell}(x) \qquad (x = (x_1, \dots, x_n)) \\
&= p_\mu(x). \qquad \square
\end{aligned}$$

29. The Frobenius characteristic map and the Murnaghan–Nakayama rule

Let R^k be the space of class functions on S_k, and let $R = \bigoplus_{k\ge 0} R^k$. Let Λ^k be the space of homogeneous symmetric functions of degree k. Recall that χ^λ is the character of S^λ and w_μ is a permutation of cycle type μ.

Definition 78. The *Frobenius characteristic map* $\mathrm{ch}\colon R \to \Lambda$ is the map
$$\mathrm{ch}(f) = \sum_{\mu \vdash k} \frac{1}{z_\mu} f(w_\mu)\, p_\mu.$$

Theorem 79.

a) $\mathrm{ch}(\chi^\lambda) = s_\lambda$.

b) ch *is a bijective linear map.*

c) ch *is an isometry with respect to the standard inner product on class functions and the inner product on Λ^k defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.*

d) ch *is an algebra isomorphism with respect to multiplication in* Λ *and the induction product*

$$\begin{aligned} R^k \times R^m &\longrightarrow R^{k+m} \\ \chi \cdot \psi &\longmapsto \operatorname{Ind}_{S_k \times S_m}^{S_{k+m}}(\chi \otimes \psi). \end{aligned}$$

Proof.

a) Consider the S_k class function $w \mapsto \operatorname{tr}_{V^{\otimes k}}(g, w)$. By Proposition 77, $\operatorname{tr}_{V^{\otimes k}}(g, w) = p_\mu(x)$, where $x_1, \ldots, x_n$ are the eigenvalues of g.
On the other hand, by Schur–Weyl duality, if $n \geq k$,

$$\operatorname{tr}_{V^{\otimes k}}(g, w) = \sum_{\lambda \vdash k} \underbrace{\operatorname{tr}_{V^\lambda}(g)}_{s_\lambda(x)} \cdot \chi^\lambda(w).$$

Equating, and applying the Frobenius characteristic map, gives

$$\sum_{\mu \vdash k} \frac{1}{z_\mu} p_\mu(x) p_\mu(y) = \sum_{\lambda \vdash n} s_\lambda(x)\ \operatorname{ch}(\chi^\lambda)(y).$$

By the Cauchy identity and linear independence of the $s_\lambda(y)$, we have $\operatorname{ch}(\chi^\lambda) = s_\lambda$.

b), c) Follow since the χ^λ and s_λ are orthonormal bases.

d) Frobenius reciprocity; see Problem 29.

□

Corollary 80 (Frobenius character formula)**.**

$$s_\lambda = \sum_{\mu \vdash k} \frac{1}{z_\mu} \chi^\lambda(w_\mu)\, p_\mu,$$

so $\chi^\lambda(w_\mu)$ *is* z_μ *times the coefficient of* p_μ *in the expansion of* s_λ *in the power sum basis.*

Proof. Apply the definition of the Frobenius characteristic (Definition 78) and Theorem 79 a). □

Proposition 81. *For all* μ *and all* k,

$$p_k s_\mu = \sum_\lambda (-1)^{h(\lambda/\mu)} s_\lambda,$$

where $\lambda/\mu := \lambda \setminus \mu$ *is a border strip of size* k.

Proof of Theorem 42 from Proposition 81. First we expand p_μ in the Schur basis. Note that

$$\operatorname{ch}(z_\mu \delta_{\mu\lambda}) = p_\mu,$$

and so

$$\langle p_\mu, p_\nu \rangle \overset{\text{isom.}}{=} \langle z_\mu \delta_{\mu\lambda}, z_\nu \delta_{\nu\lambda} \rangle = \frac{1}{k!} \sum_{w \in C_\mu \cap C_\nu} z_\mu z_\nu = z_\mu z_\nu \frac{|C_\mu|}{k!} \delta_{\mu\nu} = z_\mu \delta_{\mu\nu}.$$

By the Frobenius character formula (Corollary 80),

$$\langle s_\lambda, p_\mu \rangle = \sum_\nu \frac{1}{z_\nu} \chi^\lambda(w_\nu) \langle p_\mu, p_\nu \rangle = \chi^\lambda(w_\mu),$$

so since the s_λ are orthonormal,

$$p_\mu = \sum_\nu \chi^\nu(w_\mu)\, s_\nu.$$

Now, let $\mu' = (\mu_2, \dots, \mu_{\ell(\mu)})$. Then

$$\begin{aligned}
\chi^\lambda(w_\mu) &= \langle s_\lambda, p_{\mu_1} p_{\mu'} \rangle = \left\langle s_\lambda,\ p_{\mu_1} \sum_\nu \chi^\nu(w_{\mu'}) s_\nu \right\rangle \\
&= \sum_\nu \chi^\nu(w_{\mu'}) \langle s_\lambda, p_{\mu_1} s_\nu \rangle \\
&\overset{\substack{\text{Proposition 81} \\ \uparrow}}{=} \sum_\nu \chi^\nu(w_{\mu'}) \begin{cases} (-1)^{h(\lambda/\nu)}, & \text{if } \lambda/\nu \text{ is a border strip of size } \mu_1, \\ 0, & \text{else,} \end{cases}
\end{aligned}$$

as desired. □

Proof of Proposition 81. (See [Sta99, §7.17].) Let $n \gg 0$.

Step 1.

$$a_\alpha p_k = \sum_{j=1}^n a_{\alpha + k\epsilon_j}, \qquad \text{where } a_\alpha = \sum_{w \in S_n} (-1)^w x^{w(\alpha)}, \quad \epsilon_j = (0, \dots, \underset{j}{1}, \dots, 0).$$

Indeed,

$$\begin{aligned}
a_\alpha p_k &= \sum_{w \in S_n} (-1)^w \sum_{i=1}^n x^{w(\alpha) + k\epsilon_i} \\
&= \sum_{w \in S_n} (-1)^w \sum_{j=1}^n x^{w(\alpha) + k\epsilon_{w(j)}} \qquad (j = w^{-1}(i)) \\
&= \sum_{w \in S_n} (-1)^w \sum_{j=1}^n x^{w(\alpha + k\epsilon_j)} \\
&= \sum_{j=1}^n a_{\alpha + k\epsilon_j}.
\end{aligned}$$

Step 2: expand $p_k s_\mu$. By Step 1 and the bialternant formula,

$$p_k s_\mu = \frac{p_k a_{\mu+\rho}}{a_\rho} = \sum_{j=1}^n \frac{a_{\alpha + k\epsilon_j}}{a_\rho}, \qquad (\alpha = \mu + \rho)$$

If $\alpha_j + k = \alpha_i$ for some i, then $a_{\alpha + k\epsilon_j} = 0$. Otherwise, there is a unique rearrangement $\nu := \nu^{(j)}$ of $\alpha + k\epsilon_j$ that is a strict partition: $\nu_1 > \nu_2 > \cdots$, and

$$a_\nu = (-1)^{j-p} a_{\alpha + k\epsilon_j}, \qquad \text{where } \nu_p = (\alpha + k\epsilon_j)_j.$$

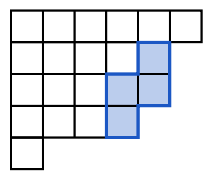
So

$$p_k s_\mu = \sum_j (-1)^{j-p} \frac{a_{\nu^{(j)}}}{a_\rho} = \sum_j (-1)^{j-p} s_{\underbrace{\nu^{(j)}-\rho}_{=:\lambda^{(j)}}}.$$

Step 3. The result now follows from the claim that $\lambda^{(j)} = \mu+$ a/the border strip starting in row j of size k, and $h(\lambda/\mu) = j - p$.

We “prove” this by example:

$$\begin{aligned} \mu &= (6,4,3,3,1) \\ \alpha = \mu + \rho &= (10,7,5,4,1) \\ \alpha + k\epsilon_j &= (10,7,5,8,1) \\ \nu &= (10,8,7,5,1) \\ \lambda = \nu - \rho &= (6,5,5,4,1) \end{aligned} \qquad \begin{aligned} & j = 5, \quad k = 4 \\ & \\ & \mu = (6,4,3,3,1) \\ & \lambda = (6,5,5,4,1) \\ & p = 2, \quad j - p = 2 = h(\lambda/\mu) \end{aligned}$$

μ 

□

Starting next week: extra topics!

Problems

Unless otherwise stated, all groups here are finite, all Lie algebras are complex and finite-dimensional, and all representations are finite-dimensional and complex.

Problem 21. Let V be a finite-dimensional vector space over a field K. Recall that an operator $A \in \mathrm{End}(V)$ is *diagonalizable* if V has a basis of eigenvectors for A, and that a set of operators $\mathcal{A}$ is *simultaneously diagonalizable* if V has a basis each of whose vectors is an eigenvector for every $A \in \mathcal{A}$.

Let $\mathcal{A} \subseteq \mathrm{End}(V)$ be a set of diagonalizable operators on V. The goal of this problem is to prove that $\mathcal{A}$ is simultaneously diagonalizable if and only if its elements pairwise commute.

a) Prove that if $\mathcal{A}$ is simultaneously diagonalizable then its elements pairwise commute.
b) Let $A, B \in \mathrm{End}(V)$ be commuting operators. For an eigenvalue λ of A, let E_λ be its λ-eigenspace, $E_\lambda = \{ v \in V \mid Av = \lambda v \}$. Prove that E_λ is a B-invariant subspace.
c) Let $A \in \mathrm{End}(V)$ and let $W \leq V$ be an A-invariant subspace. Prove that if A is diagonalizable on V then it is also diagonalizable on W.
d) Use the preceding two parts to show that if $\mathcal{A} \subseteq \mathrm{End}(V)$ is a finite collection of pairwise commuting diagonalizable operators on V, then $\mathcal{A}$ is simultaneously diagonalizable.

e) Show that the previous part holds even when $\mathcal{A}$ is infinite (with V still finite-dimensional).

Problem 22. For each of the irreducible rank 2 root systems A_2, $B_2 (= C_2)$ and G_2, verify explicitly that the Weyl group acts transitively on the set of roots of a given length.

Problem 23. Let $\mathfrak{g}$ be a complex Lie algebra. The *Killing form* is the symmetric bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by $B(X, Y) = \operatorname{tr}(\operatorname{ad}_X \circ \operatorname{ad}_Y)$.

a) Show that B is Ad-invariant: $B\big(\operatorname{Ad}_A(X), \operatorname{Ad}_A(Y)\big) = B(X, Y)$ for all $A \in G$ and $X, Y \in \mathfrak{g}$. (Hint: first show that $\operatorname{ad}_{\operatorname{Ad}_A(X)} = \operatorname{Ad}_A \circ \operatorname{ad}_X \circ \operatorname{Ad}_A^{-1}$ as operators on $\mathfrak{g}$.)
b) Compute B explicitly for $\mathfrak{sl}_2(\mathbb{C})$ with standard basis e, f, h satisfying $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

Problem 24.

a) Let X, C be $n \times n$ matrices with C invertible. Show that $e^{CXC^{-1}} = Ce^X C^{-1}$.
b) Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. For $A \in G$, let $\operatorname{Ad}_A(X) = AXA^{-1}$. Show that $\operatorname{Ad}_A \in \mathfrak{gl}(\mathfrak{g})$; that is, show that $\operatorname{Ad}_A(X) \in \mathfrak{g}$ whenever $X \in \mathfrak{g}$, and that $X \mapsto AXA^{-1}$ is a linear map.
c) Since $\operatorname{ad}_X \in \mathfrak{gl}(\mathfrak{g})$, we may define $e^{\operatorname{ad}_X} \in \mathfrak{gl}(\mathfrak{g})$ by the matrix exponential. Prove that $e^{\operatorname{ad}_X} = \operatorname{Ad}_{e^X}$ for all $X \in \mathfrak{g}$.

Problem 25. Give an example of a matrix Lie group G and a matrix X such that $e^X \in G$ but X is not in the Lie algebra $\mathfrak{g}$ of G.

Problem 26. Let G be a finite group, and equip $L^2(G) = \{\, f \mid f\colon G \to \mathbb{C} \,\}$ with the inner product $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$. By Problem 4, $L^2(G)$ under left translation is the regular representation, which by Corollary 13 decomposes as $\bigoplus_\rho V_\rho^{\oplus d_\rho}$, where $d_\rho = \dim V_\rho$ and the sum runs over all irreducible representations. The goal of this problem is to find an explicit orthonormal basis realizing this decomposition, in analogy with the Peter–Weyl theorem (Theorem 67).

For each irreducible representation (ρ, V_ρ) of G, fix a G-invariant inner product on V_ρ and an orthonormal basis $\{e_i\}_{i=1}^{d_\rho}$ with respect to it. Let $\rho_{ij}\colon G \to \mathbb{C}$ be the matrix coefficient $\rho_{ij}(g) = \langle e_i, \rho(g) e_j \rangle$; these are the matrix coefficients of Definition 65, with $\rho_{ij}(g)$ the (i, j)-entry of $\rho(g)$.

a) *(Orthogonality of matrix coefficients.)* Let ρ, π be irreducible representations of G. Show that

$$\langle \rho_{ij}, \pi_{k\ell} \rangle = \begin{cases} 0 & \text{if } \rho \not\cong \pi, \\ \dfrac{\delta_{ik}\delta_{j\ell}}{d_\rho} & \text{if } \rho = \pi. \end{cases}$$

In particular the matrix coefficients of non-isomorphic irreducibles are orthogonal, and those of a single irreducible are orthogonal to one another. (Hint: mirroring the averaging approach used a few times already, given a linear map $T\colon V_\pi \to V_\rho$, set $\widehat{T} = \frac{1}{|G|} \sum_{g \in G} \rho(g)\, T\, \pi(g)^{-1}$. Show that $\widehat{T}$ is G-equivariant, then apply Schur's lemma. Choose T to be a well-chosen rank-one map to extract the desired inner products.)
b) Using part a) and Corollary 14, show that

$$\left\{ \sqrt{d_\rho}\, \rho_{ij} \;\middle|\; \rho \text{ irreducible},\ 1 \le i, j \le d_\rho \right\}$$

is an orthonormal basis for $L^2(G)$.

Problem 27. Let $Z = (z_{ij})_{1\le i,j\le n}$ be an $n\times n$ matrix of indeterminates, and let $\mathrm{GL}_n(\mathbb{C})$ act on $\mathbb{C}[z_{ij}]$ by $(g\cdot f)(Z) = f(Zg)$. For $S\subseteq [n]$ with $|S| = k$, write

$$p_S = \det(z_{ij})_{1\le i\le k,\ j\in S}$$

for the minor of Z using rows $1,\dots,k$ and columns S. Let $N \le \mathrm{GL}_n(\mathbb{C})$ denote the subgroup of upper triangular matrices with 1's on the diagonal.

a) Show that $p_{[k]}$ is a highest weight vector of weight $\varepsilon_k = e_1 + \cdots + e_k$ — recall that ε_i is the ith fundamental weight — and that the subrepresentation generated by $p_{[k]}$ is $\operatorname{span}\{\, p_S \mid |S| = k \,\} \cong \bigwedge^k(\mathbb{C}^n)$. (Hint: use the Cauchy–Binet formula.)
b) For a partition $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n \ge 0)$ of length $\le n$, define

$$\Delta_\lambda = \prod_{k=1}^{n} p_{[k]}^{\lambda_k - \lambda_{k+1}}.$$

Show that Δ_λ is a highest weight vector of weight λ, and that the GL_n-subrepresentation of $\mathbb{C}[z_{ij}]$ it generates is isomorphic to V^λ. (Hint: use Cauchy–Binet to show that the polynomial GL_n-representation generated by Δ_λ is finite-dimensional, hence completely reducible, and apply Part 6 of the proof sketch of Theorem 57, which you may assume holds equally well for GL_n.)
c) Take $n = 3$ and $\lambda = (2,1,0)$. By part b), $V^{(2,1,0)}$ is the GL_3-representation generated by

$$\Delta_{(2,1,0)} = p_{\{1\}}\cdot p_{\{1,2\}} = z_{11}(z_{11}z_{22} - z_{12}z_{21}).$$

Write out a basis for $V^{(2,1,0)}$ explicitly as polynomials in the z_{ij}. (Hint: by the Cauchy–Binet expansion from part b), $V^{(2,1,0)}$ is contained in the span of the $3\times 3 = 9$ products $p_{\{s\}}\cdot p_{\{t,u\}}$ for $s\in[3]$, $\{t,u\}\subseteq[3]$. Find the linear relations among these nine products — you do not need a rigorous argument that you have found them all — and verify that the dimension equals $\dim V^{(2,1,0)}$.)

Problem 28. The *Kronecker coefficients* $g(\lambda,\mu,\nu)$, for $\lambda,\mu,\nu\vdash k$, are the tensor product multiplicities of S_k-representations:

$$S^\lambda \otimes S^\mu \cong \bigoplus_{\nu\vdash k} (S^\nu)^{\oplus g(\lambda,\mu,\nu)}.$$

The goal of this problem is to give a GL_n-interpretation of these coefficients via Schur–Weyl duality. Throughout, let $V = \mathbb{C}^m$ with $m\ge k$, and $W = V\otimes V = \mathbb{C}^{m^2}$.

a) Apply Schur–Weyl duality to $V^{\otimes k}\otimes V^{\otimes k}$ to show that there is an isomorphism of $\big(\mathrm{GL}(V)\times \mathrm{GL}(V)\times S_k\big)$-representations

$$V^{\otimes k}\otimes V^{\otimes k} \cong \bigoplus_{\lambda,\mu,\nu\vdash k} g(\lambda,\mu,\nu)\, V^\lambda\otimes V^\mu \otimes S^\nu,$$

where on the left S_k acts diagonally on the two $V^{\otimes k}$ factors, and on the right the two $\mathrm{GL}(V)$-factors act on V^λ and V^μ respectively.
b) Let $\mathrm{GL}(V)\times\mathrm{GL}(V)$ sit inside $\mathrm{GL}(W)$ via the Kronecker product $(A,B)\mapsto A\otimes B$. Show that the map $V^{\otimes k}\otimes V^{\otimes k} \xrightarrow{\sim} W^{\otimes k}$,

$$(v_1\otimes\cdots\otimes v_k)\otimes(w_1\otimes\cdots\otimes w_k) \longmapsto (v_1\otimes w_1)\otimes\cdots\otimes(v_k\otimes w_k),$$

is an $\big(S_k\times\mathrm{GL}(V)\times\mathrm{GL}(V)\big)$-equivariant isomorphism.

c) Using Schur–Weyl duality for $W^{\otimes k}$ and parts a) and b), conclude that as $\mathrm{GL}(V) \times \mathrm{GL}(V)$-representations,
$$\mathrm{Res}^{\mathrm{GL}(W)}_{\mathrm{GL}(V)\times\mathrm{GL}(V)} W^\nu \cong \bigoplus_{\lambda,\mu\vdash k} g(\lambda,\mu,\nu)\, V^\lambda \otimes V^\mu.$$
In particular, $g(\lambda,\mu,\nu) = \dim \mathrm{Hom}_{\mathrm{GL}(V)\times\mathrm{GL}(V)}\big(V^\lambda \otimes V^\mu,\, W^\nu\big)$.

Problem 29. Recall the Frobenius characteristic map $\mathrm{ch}\colon R = \bigoplus_k R^k \to \Lambda$ of Definition 78, where R^k is the space of class functions on S_k and the product on R is
$$\chi \cdot \psi = \mathrm{Ind}^{S_{k+m}}_{S_k\times S_m}(\chi \otimes \psi)$$
for $\chi \in R^k$, $\psi \in R^m$; and recall that $\mathrm{ch}(f) = \sum_{\mu\vdash n} z_\mu^{-1} f_\mu\, p_\mu$ for $f \in R^n$, where f_μ is the value of f on elements of cycle type μ.

a) Prove Theorem 79d): $\mathrm{ch}\colon R \to \Lambda$ is a ring homomorphism, where Λ carries its usual multiplication of symmetric functions. That is, for $\chi \in R^k$ and $\psi \in R^m$, show that $\mathrm{ch}(\chi\cdot\psi) = \mathrm{ch}(\chi)\,\mathrm{ch}(\psi)$. (Hint: one can extend the notion of a class function to allow values in Λ, or in any other $\mathbb{C}$-algebra. The Hermitian inner product of Section 5 extends coefficient-wise, and restriction, induction and Frobenius reciprocity extend to these generalized class functions by linearity: for $H \le G$, a class function χ on G, and any $\psi\colon H \to \Lambda$, $\big(\chi, \mathrm{Ind}^G_H \psi\big)_G = \big(\mathrm{Res}^G_H \chi, \psi\big)_H$. Furthermore $\mathrm{ch}(f) = (\overline{p}, f) = (p, f)$, where $p\colon S_n \to \Lambda$ sends w to p_μ for w of cycle type μ.)

b) The *Littlewood–Richardson coefficients* $c^\lambda_{\mu\nu}$, for $|\mu|+|\nu| = |\lambda|$, are defined as the tensor product multiplicities of polynomial $\mathrm{GL}_n(\mathbb{C})$-representations:
$$V^\mu \otimes V^\nu \cong \bigoplus_\lambda (V^\lambda)^{\oplus c^\lambda_{\mu\nu}}.$$
Since the character of V^λ is s_λ, these are also the structure constants for the Schur basis: $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$. Using part a), show that as S_n-representations
$$\mathrm{Ind}^{S_n}_{S_k\times S_m}(S^\mu \otimes S^\nu) \cong \bigoplus_\lambda (S^\lambda)^{\oplus c^\lambda_{\mu\nu}}, \qquad \mathrm{Res}^{S_n}_{S_k\times S_m} S^\lambda \cong \bigoplus_{\mu,\nu} (S^\mu\otimes S^\nu)^{\oplus c^\lambda_{\mu\nu}},$$
where $k = |\mu|$, $m = |\nu|$ and $n = k+m$. (do not confuse the internal and external tensor products; the latter occurs when we have a direct product of groups)

PART IV

Further topics

Invariant theory

30. Definitions and finiteness

The references for this topic are [GW09, Ch. 5] and [KP96].

Let V be a finite-dimensional $\mathbb{C}$-vector space and let $G \to \mathrm{GL}(V)$ be a representation, where G is a finite group or a $\mathbb{C}$-matrix Lie group, e.g. GL_n, SL_n, $\mathrm{O}(n)$, Sp_{2n}.

This induces a G-representation on the coordinate ring $\mathbb{C}[V] := \mathrm{Sym}(V^*)$ via

$$(g \cdot f)(v) = f(g^{-1}v), \qquad g \in G,\ f \in \mathbb{C}[V],\ v \in V.$$

If f is homogeneous of degree d, so is $g \cdot f$.

Definition 82. $f \in \mathbb{C}[V]$ is an *invariant* if $g \cdot f = f$ for all $g \in G$. The *ring of invariants* is the subalgebra of invariants

$$\mathbb{C}[V]^G := \{ f \in \mathbb{C}[V] \mid g \cdot f = f\ \forall g \in G \}.$$

Remark 10. More generally, if X is an affine variety and G acts on X, then we similarly have a G-action on $\mathbb{C}[X]$, which is sometimes a representation.

Examples:

a) $G = S_n$, $V = \mathbb{C}^n = \mathrm{span}\{v_1, \dots, v_n\}$, with S_n acting on V by permuting coordinates: $w \cdot v_i = v_{w^{-1}(i)}$. Then $\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]$ where $x_i(v_j) = \delta_{ij}$, and the ring of invariants is

$$\mathbb{C}[x_1, \dots, x_n]^{S_n} = \begin{array}{c}\text{ring of symmetric}\\ \text{polynomials in } x_1, \dots, x_n\end{array} \cong \mathbb{C}[e_1, \dots, e_n],$$

the e_i being the elementary symmetric polynomials, by the Fundamental Theorem of Symmetric Functions (see also Section 28).

b) $G = \mathrm{SL}_n$, $V = \mathrm{Mat}_n(\mathbb{C})$, with SL_n acting on V by left multiplication, $g \cdot X := gX$. We claim

$$\mathbb{C}[\mathrm{Mat}_n]^{\mathrm{SL}_n} = \mathbb{C}[\det], \qquad \text{where } \det = \sum_w (-1)^w z_{1,w(1)} \cdots z_{n,w(n)} \in \mathbb{C}[V].$$

On one hand, $\det(gX) = \det(g)\det(X) = \det(X)$, so $\det \in \mathbb{C}[\mathrm{Mat}_n]^{\mathrm{SL}_n}$.
Conversely, if $f \in \mathbb{C}[\mathrm{Mat}]^{\mathrm{SL}_n}$, then define $p \in \mathbb{C}[t]$ by

$$p(t) := f\left(\begin{bmatrix} t & & \\ & \ddots & \\ & & 1 \end{bmatrix}\right).$$

Then if $A \in \mathrm{GL}_n$ with $\det A = t$,

$$f(A) = f\Big(\underset{\in \mathrm{SL}_n}{g}\begin{bmatrix} t & & \\ & \ddots & \\ & & 1 \end{bmatrix}\Big) = f\left(\begin{bmatrix} t & & \\ & \ddots & \\ & & 1 \end{bmatrix}\right) = p(t).$$

Since GL_n is (Zariski) dense in Mat_n, $f(A) = p(\det A)$ for all A by continuity.

c) $G = \mathrm{O}(n) = \{ g \in \mathrm{GL}_n \mid gg^T = I \}$, $V = \mathbb{C}^n$, acting by left multiplication. Then

$$\mathbb{C}[V]^{\mathrm{O}(n)} = \mathbb{C}[\underbrace{x_1^2 + x_2^2 + \cdots + x_n^2}_{N^2}], \qquad N^2(v) = \|v\|^2 .$$

d) $G = \mathrm{GL}_n$, $V = \mathrm{Mat}_n$, with the action by conjugation $g \cdot X := gXg^{-1}$. The characteristic polynomial is

$$\mathrm{ch}_X(t) = \det(tI - X) = t^n + \sum_{i=1}^n (-1)^i e_i(X) t^{n-i},$$

where $e_i = e_i(X)$ is the ith elementary symmetric polynomial in the eigenvalues of X. Then conjugation preserves $\mathrm{ch}_X(t)$, so

$$\mathbb{C}[e_1, \dots, e_n] \subseteq \mathbb{C}[\mathrm{Mat}_n]^{\mathrm{GL}_n}.$$

On the other hand, if $f \in \mathbb{C}[\mathrm{Mat}_n]^{\mathrm{GL}_n}$, then if $D = \{\mathrm{diag}(a_1, \dots, a_n)\}$, $f|_D$ is symmetric in $a_1, \dots, a_n$, and a similar continuity argument to the above shows that $f \in \mathbb{C}[e_1, \dots, e_n]$. Also note that $\mathrm{tr}_k \colon X \mapsto \mathrm{tr}(X^k)$ is the kth power sum symmetric function in the eigenvalues of X, so

$$\mathbb{C}[\mathrm{Mat}_n]^{\mathrm{GL}_n} \cong \mathbb{C}[\mathrm{tr}_1, \mathrm{tr}_2, \dots, \mathrm{tr}_n].$$

Other examples: [KP96].

Questions.

Q1 Is $\mathbb{C}[V]^G$ finitely generated?

Q2 What is a set of generators? — *First Fundamental Theorem* (FFT)

Q3 What are the relations between these generators? — *Second Fundamental Theorem* (SFT)

Theorem 83 (Hilbert's Finiteness Theorem). *If G has complete reducibility (reductive), then $\mathbb{C}[V]^G$ is finitely generated.*

(In general, the answer to Q1 is "no".)

Definition 84. A *Reynolds operator* is a linear map $R \colon \mathbb{C}[V] \to \mathbb{C}[V]^G$ satisfying

- $R(f) = f$ for all $f \in \mathbb{C}[V]^G$;
- $R(f \cdot \psi) = R(f) \cdot \psi$ for all $f \in \mathbb{C}[V]$, $\psi \in \mathbb{C}[V]^G$.

R always exists if G is reductive (the averaging operator if G is compact).

Proof sketch of Theorem 83. Write $J = \mathbb{C}[V]^G$ and $J_+ = \{ f \in J \mid f(0) = 0 \}$. Let I be the ideal $I = (J_+) \subseteq \mathbb{C}[V]$.

By the Hilbert basis theorem, $\mathbb{C}[V]$ is Noetherian, so I is a finitely generated ideal, $I = (\phi_1, \dots, \phi_m)$ with $\phi_i \in J_+$ homogeneous. We claim that $J = \mathbb{C}[\phi_1, \dots, \phi_m]$.

Let $\phi \in J_+$ be homogeneous of degree $d > 0$, and induct on d. Since $\phi \in J_+ \subseteq I$, write

$$\phi = \sum_i f_i \phi_i \qquad \text{with } f_i \in \mathbb{C}[V].$$

Apply the Reynolds operator:

$$\phi = R(\phi) = \sum_i R(f_i\phi_i) = \sum_i \underset{\in J}{R(f_i)}\, \phi_i.$$

We have $\deg R(f_i) = d - \deg \phi_i < d$, so by induction each $R(f_i) \in \mathbb{C}[\phi_1, \dots, \phi_m]$; hence so is ϕ. $\square$

31. The first fundamental theorem for classical groups

Let $V = \mathbb{C}^n$. Both V and V^* are GL_n-representations:

$$g \cdot v := gv \quad (v \in V), \qquad g \cdot \varphi := \varphi \circ g^{-1} \quad (\varphi \in V^*).$$

So GL_n acts on $W = V^p \oplus (V^*)^q = V \oplus \cdots \oplus V \oplus V^* \oplus \cdots \oplus V^*$ by

$$g \cdot \underbrace{(v_1, \dots, v_p, \varphi_1, \dots, \varphi_q)}_{W} = \big(gv_1, \dots, gv_p, \varphi_1 \circ g^{-1}, \dots, \varphi_q \circ g^{-1}\big).$$

For $1 \le i \le p$, $1 \le j \le q$, the *contraction* is the map

$$(i|j)\colon W \longrightarrow \mathbb{C}, \qquad w \longmapsto \varphi_j(v_i).$$

We have $(i|j) \in \mathbb{C}[W]^{\mathrm{GL}_n}$, since

$$g \cdot (i|j)(w) = (i|j)(g^{-1}w) = (\varphi_j \circ g)(g^{-1}w) = \varphi_j(v_i) = (i|j)(w).$$

Theorem 85 (FFT for GL_n). *$\mathbb{C}[W]^{\mathrm{GL}_n}$ is generated as a $\mathbb{C}$-algebra by the contractions $(i|j)$.*

We will start with the special case of *multilinear* functions:

$$f\big(\lambda_1 v_1, \dots, \lambda_p v_p, \mu_1\varphi_1, \dots, \mu_q\varphi_q\big) = \lambda_1 \cdots \lambda_p\, \mu_1 \cdots \mu_q\, f(w),$$

which are elements in $\big(V^{\otimes p} \otimes (V^*)^{\otimes q}\big)^*$.

Lemma 86. *The multilinear elements of $\mathbb{C}[W]^{\mathrm{GL}_n}$ satisfy:*

a) If $p \ne q$, the only such function is 0.

b) If $p = q$, they are spanned by $\{\, f_\sigma \mid \sigma \in S_p \,\}$, where

$$f_\sigma(w) = \prod_{i=1}^{p} \varphi_{\sigma(i)}(v_i), \qquad \textit{i.e.} \quad f_\sigma = \big(\sigma(1)\big|1\big) \cdots \big(\sigma(p)\big|p\big).$$

Proof. Let $g = \lambda I \in \mathrm{GL}_n$, $\lambda \in \mathbb{C}^\times$. Then if $f \ne 0$ is multilinear,

$$(g \cdot f)(w) = f(\lambda^{-1}w) = \lambda^{q-p} f(w).$$

GL_n-invariance thus requires $p = q$.

If $p = q$, then the multilinear functions on W are $(U \otimes U^*)^* \cong \mathrm{End}(U)$, where $U = V^{\otimes p}$ and $\mathrm{GL}_n \le \mathrm{GL}(U)$ acts on $\mathrm{End}(U)$ by conjugation. The isomorphism is

$$\alpha\colon\ \mathrm{End}(U) \longrightarrow (U \otimes U^*)^*, \qquad \alpha(A)(u \otimes \phi) = \phi(Au),$$

which is $\mathrm{GL}(U)$-equivariant, hence GL_n-equivariant.

Now, if $f \in \mathbb{C}[W]^{\mathrm{GL}_n}$, by Proposition 72,

$$f \in \mathrm{End}_{\mathrm{GL}_n}\big(V^{\otimes p}\big) = \mathrm{span}\,\{\, \sigma \mid \sigma \in S_p \,\} \subseteq \mathrm{End}\big(V^{\otimes p}\big).$$

Finally, if $\sigma \in \mathrm{End}(V^{\otimes p})$, then

$$\begin{aligned}
&\alpha(\sigma)(v_1 \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \varphi_p) \\
&\quad = (\varphi_1 \otimes \cdots \otimes \varphi_p)\big(\sigma(v_1 \otimes \cdots \otimes v_p)\big) \\
&\quad = (\varphi_1 \otimes \cdots \otimes \varphi_p)\big(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)}\big) \\
&\quad = \big(\varphi_1 \big| v_{\sigma^{-1}(1)}\big) \cdots \big(\varphi_p \big| v_{\sigma^{-1}(p)}\big) \\
&\quad = f_\sigma(v_1 \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \varphi_p). \qquad \square
\end{aligned}$$

Proof sketch of Theorem 85. A function f is *multihomogeneous* if there exist $d_1, \ldots, d_p, e_1, \ldots, e_q$ such that

$$f\big(\lambda_1 v_1, \ldots, \lambda_p v_p, \mu_1 \varphi_1, \ldots, \mu_q \varphi_q\big) = \lambda_1^{d_1} \cdots \lambda_p^{d_p}\, \mu_1^{e_1} \cdots \mu_q^{e_q}\, f(w).$$

Any $f \in \mathbb{C}[W]$ is the sum of its multihomogeneous components.

Any multihomogeneous $f \in \mathbb{C}[W]^{\mathrm{GL}_n}$ of degrees $d_1, \ldots, d_p, e_1, \ldots, e_q$, with $D = \sum d_i$, $E = \sum e_i$, has a unique *polarization*, a G-invariant multilinear function

$$\widehat{f} \in \mathbb{C}\big[V^D \oplus (V^*)^E\big]^{\mathrm{GL}_n}$$

that satisfies the *restitution*

$$\widehat{f}\big(\underbrace{v_1, \ldots, v_1}_{d_1}, \ldots, \underbrace{v_p, \ldots, v_p}_{d_p}, \underbrace{\varphi_1, \ldots, \varphi_1}_{e_1}, \ldots, \underbrace{\varphi_q, \ldots, \varphi_q}_{e_q}\big) = f(v_1, \ldots, v_p, \varphi_1, \ldots, \varphi_q).$$

By Lemma 86, $\widehat{f}$ is generated by polynomials in the contractions $(i|j)$, so by restitution, so is f. $\square$

Other FFTs.

Theorem 87.

a) *Let $V = \mathbb{C}^n$ and let $(v, w) := v \cdot w$. Then $\mathbb{C}[V^k]^{\mathrm{O}(n)}$ is generated by the quadratic polynomials (v_i, v_j), $1 \le i \le j \le k$.*

b) *Let $V = \mathbb{C}^{2n}$ and let $\langle v, w\rangle := v^T J w$, with J as in Theorem 54's symplectic case. Then $\mathbb{C}[V^k]^{\mathrm{Sp}_{2n}(\mathbb{C})}$ is generated by the quadratic polynomials $\langle v_i, v_j\rangle$, $1 \le i < j \le k$.*

c) *Let GL_n act on $\mathrm{End}(\mathbb{C}^n)^m$ by simultaneous conjugation:*

$$g \cdot (A_1, \ldots, A_m) = \big(gA_1g^{-1}, \ldots, gA_mg^{-1}\big).$$

Then $\mathbb{C}\big[\mathrm{End}(V)^m\big]^{\mathrm{GL}_n}$ is generated by elements

$$\mathrm{tr}\big(A_{i_1} \cdots A_{i_r}\big), \qquad 1 \le i_j \le n, \quad r \le n^2.$$

(A special case of Theorem 85.)

32. The second fundamental theorem, and finite groups

Recall $W = V^p \oplus (V^*)^q$ with $\mathrm{GL}_n = \mathrm{GL}(V)$ acting by

$$g \cdot \underbrace{(v_1, \dots, v_p, \varphi_1, \dots, \varphi_q)}_{w=(v,\varphi)} = \big(gv_1, \dots, gv_p, \varphi_1 \circ g^{-1}, \dots, \varphi_q \circ g^{-1}\big).$$

Identify

$$V^p \cong \mathrm{Hom}(\mathbb{C}^p, V), \quad e_i \mapsto v_i, \qquad (V^*)^q \cong \mathrm{Hom}(V, \mathbb{C}^q), \quad v \mapsto \big(\varphi_1(v), \dots, \varphi_q(v)\big),$$

and define

$$\mu\colon W \longrightarrow \mathrm{Mat}_{q,p}(\mathbb{C}), \qquad \mu(v,\varphi) = \varphi \circ v \colon \mathbb{C}^p \xrightarrow{v} V \xrightarrow{\varphi} \mathbb{C}^q \ \in \ \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q),$$

i.e.

$$\mu(v,\varphi) = \begin{bmatrix} \varphi_1(v_1) & \cdots & \varphi_1(v_p) \\ \varphi_2(v_1) & \ddots & \varphi_2(v_p) \\ \vdots & & \vdots \\ \varphi_q(v_1) & \cdots & \varphi_q(v_p) \end{bmatrix}.$$

The pullback is

$$\mu^*\colon \mathbb{C}[\mathrm{Mat}_{q,p}] \longrightarrow \mathbb{C}[W], \qquad f \longmapsto f \circ \mu,$$

and $\mu^*(z_{ij}) = z_{ij} \circ \mu = (j|i)$, where z_{ij} is the coordinate function. By the FFT (Theorem 85), $\mu^*\colon \mathbb{C}[\mathrm{Mat}_{q,p}] \to \mathbb{C}[W]^{\mathrm{GL}_n}$ is surjective.

Theorem 88 (SFT for GL_n)**.** *The kernel of μ^* is the ideal $I_{n+1} \subseteq \mathbb{C}[\mathrm{Mat}_{q,p}]$ generated by $(n+1) \times (n+1)$ minors. In particular, if $n \geq \min(q,p)$, then the contractions are algebraically independent.*

We will need:

Theorem 89. *I_{n+1} is a radical ideal, corresponding to the determinantal variety*

$$D_{q,p}^{(n)} = \{\, Z \in \mathrm{Mat}_{q,p} \mid \mathrm{rank}\, Z \leq n \,\}.$$

Proof of Theorem 88. We claim that $\mathrm{im}\,\mu = D_{q,p}^{(n)}$. Since $\mu(v,\varphi)$ factors through V, its rank must be $\leq n$. Conversely, if $Z \in \mathrm{Mat}_{q,p}$ has rank $\leq n$, then it can be seen to factor into $Z = XY$ with $X \in \mathrm{Mat}_{q,n}$, $Y \in \mathrm{Mat}_{n,p}$, so $Z \in \mathrm{im}\,\mu$.

Indeed, if Z has rank $r \leq n$, we can write

$$Z = U \underbrace{\begin{bmatrix} I_r & \\ & 0 \end{bmatrix}}_{q \times p} V, \qquad U \in \mathrm{GL}_q,\ V \in \mathrm{GL}_p,$$

and then take

$$X = U \underbrace{\begin{bmatrix} I_r & \\ & 0 \end{bmatrix}}_{q \times n} \in \mathrm{Mat}_{q,n}, \qquad Y = \underbrace{\begin{bmatrix} I_r & \\ & 0 \end{bmatrix}}_{n \times p} V \in \mathrm{Mat}_{n,p}.$$

Now, $f \in \ker(\mu^*)$ if and only if $f \circ \mu = 0$, i.e. f vanishes on $\operatorname{im}\mu = D^{(n)}_{q,p}$. Thus, by the Nullstellensatz,

$$\ker(\mu^*) = I\big(D^{(n)}_{q,p}\big) = \sqrt{I_{n+1}},$$

and by Theorem 89, $\sqrt{I_{n+1}} = I_{n+1}$. □

Corollary 90. *The invariant ring of* GL_n *acting on* $V^p \oplus (V^*)^q$ *is isomorphic to the coordinate ring of the determinantal variety:*

$$\mathbb{C}\big[V^p \oplus (V^*)^q\big]^{\mathrm{GL}_n} \cong \mathbb{C}[\mathrm{Mat}_{q,p}]/I_{n+1} = \mathbb{C}\big[D^{(n)}_{q,p}\big].$$

Invariant theory of finite groups.

For a graded $\mathbb{C}$-algebra $R = \bigoplus_{d>0} R_d$ with $\dim R_d < \infty$, the *Hilbert series* is

$$H(R,t) = \sum_{d\geq 0} (\dim R_d)\, t^d.$$

Theorem 91 (Molien's formula)**.** *Let G be a finite group, and let (ρ, V) be a representation. Then*

$$H\big(\mathbb{C}[V]^G, t\big) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det\big(I - t\rho(g)\big)}.$$

Proof. Since $\mathbb{C}[V] = \mathrm{Sym}(V^*)$, the degree-$d$ piece is $\mathbb{C}[V]_d = \mathrm{Sym}^d(V^*)$, and each $g \in G$ acts on V^* by $\varphi \mapsto \varphi \circ \rho(g)^{-1}$ (the contragredient representation).

By Proposition 9 (or character orthogonality), the multiplicity of the trivial representation in $\mathrm{Sym}^d(V^*)$ is

$$\dim \mathrm{Sym}^d(V^*)^G = \frac{1}{|G|}\sum_{g\in G}\chi_{\mathrm{Sym}^d(V^*)}(g).$$

If g acts on V^* with eigenvalues $\alpha_1, \dots, \alpha_n$, then

$$\chi_{\mathrm{Sym}^d(V^*)}(g) = \sum_{|\beta|=d}\alpha_1^{\beta_1}\cdots\alpha_n^{\beta_n} = h_d(\alpha_1,\dots,\alpha_n).$$

Summing over all d, we get

$$\sum_{d\geq 0} h_d(\alpha_1,\dots,\alpha_n)t^d = \prod_{i=1}^n \frac{1}{1-\alpha_i t} = \frac{1}{\det\big(I - t\, g|_{V^*}\big)},$$

the α_i being the eigenvalues of $g|_{V^*}$ and $\rho^*(g) = \rho(g^{-1})^T$, which does not affect the determinant. Summing over all $g \in G$,

$$H\big(\mathbb{C}[V]^G, t\big) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det\big(I - t\rho(g^{-1})\big)} = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det\big(I - t\rho(g)\big)},$$

replacing g with g^{-1}. □

Example 22. $G = \mathbb{Z}/n\mathbb{Z}$ acting on $\mathbb{C}$ by $a \mapsto e^{2\pi i a/n} =: \zeta^a$. By Molien's formula,

$$H\big(\mathbb{C}[x]^G, t\big) = \frac{1}{n}\sum_{a=0}^{n-1} \frac{1}{1-\zeta^a t} = \frac{1}{1-t^n}.$$

Direct computation shows that $\mathbb{C}[x]^G = \mathbb{C}[x^n]$. ✓

Theorem 92 (Chevalley–Shephard–Todd)**.** *Let G be a finite subgroup of $\mathrm{GL}(V)$, $V \cong \mathbb{C}^n$. Then $\mathbb{C}[V]^G$ is a polynomial ring if and only if G is generated by* pseudoreflections*, non-identity elements fixing a codimension-1 subspace of V.*

If this holds,

$$\mathbb{C}[V]^G \cong \mathbb{C}[\underbrace{f_1, \ldots, f_k}_{\text{alg. indep.}}],$$

we have $k = n$, $|G| = d_1 \cdots d_n$ where $d_i := \deg(f_i)$, and

$$H\big(\mathbb{C}[V]^G, t\big) = \prod_{i=1}^{n} \frac{1}{1-t^{d_i}}.$$

Examples:

a) $G = S_n = \big\langle s_i = (i, i+1)\big\rangle$, $V = \mathbb{C}^n$ (the standard representation). Then $\mathbb{C}[V]^G = \mathbb{C}[e_1, \ldots, e_n]$ with $\deg e_i = i$, so $|G| = 1 \cdots n$ and

$$H\big(\mathbb{C}[V]^G, t\big) = \prod_{i=1}^{n} \frac{1}{1-t^i}.$$

b) $G = \mathbb{Z}/2\mathbb{Z} = \{I, -I\}$, $V = \mathbb{C}^2$. Here $-I$ fixes only $\{0\}$, so it is not a pseudoreflection. So Chevalley–Shephard–Todd implies that $\mathbb{C}[V]^G$ is not a polynomial ring.
Invariants: polynomials in $\mathbb{C}[x, y]$ with all terms of even total degree. So

$$\mathbb{C}[V]^G \cong \mathbb{C}[\underbrace{x^2}_{a}, \underbrace{xy}_{b}, \underbrace{y^2}_{c}] \cong \mathbb{C}[a, b, c]/(ac - b^2), \qquad \text{(not a polynomial ring!)}$$

and by Molien's formula,

$$H\big(\mathbb{C}[V]^G, t\big) = \frac{1}{2}\left[\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2}\right] = \frac{1+t^2}{(1-t^2)^2}.$$

Quiver representations

33. Definitions and examples

The references for this topic are [Sch14] and [EGH+11, Ch. 6].

Definition 93.

a) A *quiver* $Q = (Q_0, Q_1, s, t)$ is a directed graph with

$$\begin{array}{lll} Q_0\colon \text{vertices} & s\colon Q_1 \to Q_0 \text{ “source”} & s(\alpha) \xrightarrow{\alpha} t(\alpha) \\ Q_1\colon \text{arrows} & t\colon Q_1 \to Q_0 \text{ “target”} & \end{array}$$

b) A *representation* of Q is an assignment of
   - to each vertex w, a vector space V_w;
   - to each arrow $v \xrightarrow{\alpha} w$, a linear map $V_\alpha\colon V_v \to V_w$.

   (We will restrict to finite quivers and finite-dimensional $\mathbb{C}$-vector spaces.)

c) A *morphism* $V \to W$ of Q-representations is a collection of maps

$$\varphi_v\colon V_v \to W_v, \qquad v \in Q_0,$$

that commute with the linear maps:

$$\begin{tikzcd} V_v \arrow[r, "V_\alpha"] \arrow[d, "\varphi_v"'] & V_w \arrow[d, "\varphi_w"] \\ W_v \arrow[r, "W_\alpha"] & W_w \end{tikzcd}$$

Isomorphism, subrepresentation, quotient representation and direct sum are defined similarly as for groups, Lie algebras, etc.

d) A Q-representation V is *simple* (or *irreducible*) if its only subrepresentations are V and $\{0\}$ $(V_v = \{0\}, V_\alpha = 0)$. V is *indecomposable* if $V \cong V' \oplus V''$ implies $V' = 0$ or $V'' = 0$.

By the Krull–Schmidt Theorem, every finite-dimensional quiver representation can be written as a direct sum of indecomposables, unique up to isomorphism and ordering.

Examples:

a) *Vertex*: $Q = \bullet$. Representations: $\mathbb{C}^n$. Only one indecomposable: $\mathbb{C}$ (also simple).

b) *Arrow*: $Q = \bullet \longrightarrow \bullet$. Representations: $\mathbb{C}^m \xrightarrow{A} \mathbb{C}^n$, $A \in \mathrm{Mat}_{n,m}$.
   Choosing bases of $\mathbb{C}^m$, $\mathbb{C}^n$, we can write $A = I_r \oplus 0$, where $r := \operatorname{rank} A$:

$$\begin{array}{ccc} \mathbb{C}^r & \xrightarrow{I} & \mathbb{C}^r \\ \oplus & & \oplus \\ \mathbb{C}^{m-r} & & \mathbb{C}^{n-r} \end{array} \qquad \begin{array}{l} (\mathbb{C}^{m-r} = \ker A) \\ (\mathbb{C}^{n-r} = \operatorname{coker} A) \end{array}$$

Indecomposables:

$$\underbrace{\mathbb{C} \longrightarrow 0}_{S(1)}, \qquad \underbrace{0 \longrightarrow \mathbb{C}}_{S(2)}, \qquad \underbrace{\mathbb{C} \xrightarrow{1} \mathbb{C}}_{P(1)} \quad \text{(not simple!)}$$

Morphisms:

$$\begin{tikzcd}
S(2) & 0 \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, "1"] \\
P(1) & \mathbb{C} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{C} \arrow[d] \\
S(1) & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

c) *Loop*: $Q = \bullet \circlearrowleft$. Representations: $\mathbb{C}^m \circlearrowleft A$, $A \in \mathrm{End}(\mathbb{C}^m)$.
Indecomposables $\leftrightarrow$ Jordan blocks

$$J_{\lambda,n} = \underbrace{\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}}_{n}, \qquad \lambda \in \mathbb{C}, \quad n \geq 1.$$

d) *Kronecker quiver* $\bullet \rightrightarrows \bullet$. Representations: $\begin{tikzcd} V \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] & W \end{tikzcd}$.

The indecomposables with $\dim V = \dim W = 1$ are

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & \mathbb{C} \end{tikzcd} \qquad \begin{array}{l} a, b \text{ not both } 0 \\ \text{up to mutual scalar} \end{array} \qquad \text{i.e. points } [a : b] \text{ in } \mathbb{P}^1(\mathbb{C}).$$

e) A_n *Dynkin diagram*, any orientation, e.g.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longleftarrow \bullet$$

Indecomposables correspond to intervals:

$$V(i,j)\colon \quad \overset{1}{0} \;—\; 0 \;—\; \cdots \;—\; 0 \;—\; \overset{i}{\mathbb{C}} \xrightarrow{1} \mathbb{C} \;—\; \cdots \xrightarrow{1} \overset{j}{\mathbb{C}} \;—\; 0 \;—\; \cdots \;—\; \overset{n}{0}$$

f) D_4 *Dynkin diagram*

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet & \bullet \arrow[l] \\
& \bullet \arrow[u] &
\end{tikzcd}$$

Simples:

$$\begin{tikzcd}
\mathbb{C} \arrow[r] & 0 & 0 \arrow[l] \\
& 0 \arrow[u] &
\end{tikzcd}
\quad
\begin{tikzcd}
0 \arrow[r] & 0 & \mathbb{C} \arrow[l] \\
& 0 \arrow[u] &
\end{tikzcd}
\quad
\begin{tikzcd}
0 \arrow[r] & 0 & 0 \arrow[l] \\
& \mathbb{C} \arrow[u] &
\end{tikzcd}
\quad
\begin{tikzcd}
0 \arrow[r] & \mathbb{C} & 0 \arrow[l] \\
& 0 \arrow[u] &
\end{tikzcd}$$

Other indecomposables:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \\ & 0 \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} & 0 \arrow[l] \\ & \mathbb{C} \arrow[u, "1"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \\ & 0 \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} & 0 \arrow[l] \\ & \mathbb{C} \arrow[u, "1"'] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \\ & \mathbb{C} \arrow[u, "1"'] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \\ & \mathbb{C} \arrow[u, "1"'] & \end{tikzcd} \quad \begin{tikzcd} \mathbb{C} \arrow[r, "\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]"] & \mathbb{C}^2 & \mathbb{C} \arrow[l, "\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"'] \\ & \mathbb{C} \arrow[u, "\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]"'] & \end{tikzcd}$$

Three subspaces:

$$\begin{tikzcd} U \arrow[dr, hook] & & V \arrow[dl, hook'] \\ & U+V+W & \\ & W \arrow[u, hook] & \end{tikzcd}$$

$$\begin{aligned} \dim(U+V+W) &= \dim U + \dim V + \dim W \\ &\quad - \dim(U\cap V) - \dim(U\cap W) - \dim(V\cap W) \\ &\quad + \dim(U\cap V\cap W) + \# \left(\begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{C}^2 & \mathbb{C} \arrow[l] \\ & \mathbb{C} \arrow[u] & \end{tikzcd} \right) \end{aligned}$$

(mathoverflow/23478: "Most common mathematical false belief")

34. Gabriel's theorem, path algebras, bound quiver algebras

Theorem 94 (Gabriel's Theorem)**.** *Q has finitely many indecomposables if and only if it is a union of orientations of Dynkin diagrams of type A_n, D_n or E_n. In this case, indecomposables are in bijection with positive roots.*

Proof. Uses "reflection functors"; see [EGH+11, §§6.6–6.8], [Sch14, Ch. 3, Ch. 8]. □

A_n: $\quad V(i,j) \longleftrightarrow e_i - e_j \quad (i<j).$

D_4: simple roots

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4, \quad \alpha_4 = e_3 + e_4.$$

Other positive roots

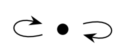
$$\begin{aligned} \alpha_1 + \alpha_2 = e_1 - e_3, \quad \alpha_2 + \alpha_3 = e_2 - e_4, \quad \alpha_2 + \alpha_4 = e_2 + e_4 \\ \alpha_1 + \alpha_2 + \alpha_3 &= e_1 - e_4 \\ \alpha_1 + \alpha_2 + \alpha_4 &= e_1 + e_4 \\ \alpha_2 + \alpha_3 + \alpha_4 &= e_2 + e_3 \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= e_1 + e_3 \\ \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 &= e_1 + e_2 \end{aligned}$$

Matching these against the D_4 indecomposables listed in Definition 93 f), and writing a representation by its dimension vector $(\dim V_1, \dim V_2, \dim V_3, \dim V_4)$ with vertex 2 the centre and vertex 4 the bottom:

root	dimension vector	root	dimension vector
α_1	$(1,0,0,0)$	$\alpha_1+\alpha_2+\alpha_3$	$(1,1,1,0)$
α_3	$(0,0,1,0)$	$\alpha_1+\alpha_2+\alpha_4$	$(1,1,0,1)$
α_4	$(0,0,0,1)$	$\alpha_2+\alpha_3+\alpha_4$	$(0,1,1,1)$
α_2	$(0,1,0,0)$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$	$(1,1,1,1)$
$\alpha_1+\alpha_2$	$(1,1,0,0)$	$\alpha_1+2\alpha_2+\alpha_3+\alpha_4$	$(1,2,1,1)$
$\alpha_2+\alpha_3$	$(0,1,1,0)$		
$\alpha_2+\alpha_4$	$(0,1,0,1)$		

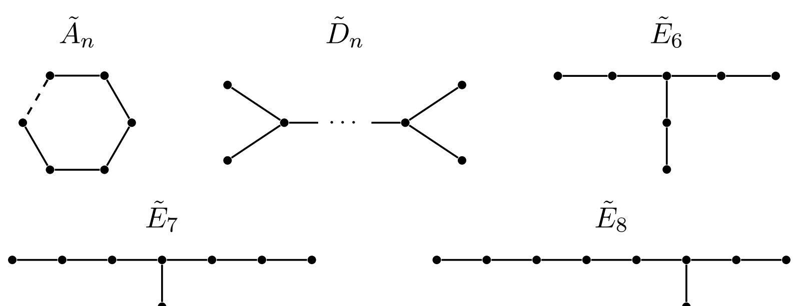
Drozd: non-finite-type quivers are either *tame* or *wild*.

- *Tame*: finitely many one-parameter families of indecomposables plus finitely many exceptions.
- *Wild*: contains the problem of classifying representations of



All wild problems contain all other wild problems.

Tame quivers are orientations of these graphs:



Definition 95.

a) The *path algebra* of a quiver Q is the $\mathbb{C}$-algebra $\mathbb{C}Q$ spanned by all paths in Q, including the "lazy paths" $\{ e_v \mid v \in Q_0 \}$, with relations

$$p \cdot q = \begin{cases} p \circ q & \text{if } q \text{ ends where } p \text{ begins,} \\ 0 & \text{else,} \end{cases} \qquad (\circ \text{ is concatenation})$$

The category of finite-dimensional left $\mathbb{C}Q$-modules is equivalent to the category of finite-dimensional Q-representations via

$$\begin{aligned} M &\longmapsto V, \qquad \text{where } V_v = e_v M \text{ and, for } v \xrightarrow{\alpha} w, \\ &\qquad V_\alpha \colon e_v M \to e_w M, \quad e_v m \mapsto \alpha m; \\ V &\longmapsto M = \bigoplus_{v \in Q_0} V_v, \qquad \mathbb{C}Q\text{-action by the maps } V_\alpha. \end{aligned}$$

Example 23. $Q = \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet}$, and

$$\mathbb{C}Q = \operatorname{span}\{e_1, e_2, e_3, \alpha, \beta, \beta\alpha\} \cong \begin{bmatrix} \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \end{bmatrix} \quad \begin{bmatrix} e_1 & & \\ \alpha & e_2 & \\ \beta\alpha & \beta & e_3 \end{bmatrix}$$

b) Let $R_Q \subseteq \mathbb{C}Q$ be the two-sided ideal generated by all arrows. A two-sided ideal $I \subseteq \mathbb{C}Q$ is *admissible* if $R_Q^m \subseteq I \subseteq R_Q^2$ for some $m \geq 2$. A *bound quiver algebra* is a quotient $\mathbb{C}Q/I$ where I is admissible.

Example 24. For $\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet}$, $I = (\beta\alpha)$ is admissible, and

$$\mathbb{C}Q/I = \operatorname{span}\{e_1, e_2, e_3, \alpha, \beta\}.$$

The quotient loses the indecomposable $\mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C}$, since $\beta\alpha = 0$.

Theorem 96. *Let A be a finite-dimensional $\mathbb{C}$-algebra. There exists a bound quiver algebra $\mathbb{C}Q/I$ such that there is an equivalence of categories*

$$A\text{-}mod \;\cong\; \mathbb{C}Q/I\text{-}mod.$$

Examples:

a) $A = \mathbb{C}[G]$, G a finite group, with $V_1, \dots, V_k$ the irreps of G. Then

$$Q = \overset{1}{\bullet} \qquad \overset{2}{\bullet} \qquad \cdots \qquad \overset{k}{\bullet} \quad \text{(no arrows)} \qquad I = 0,$$

and Q-representations are direct sums of $S(1), \dots, S(k)$.

b) $A = \mathbb{C}[x]/(x^n)$. One simple A-module: $V = \mathbb{C}$, where x acts by 0. Indecomposables: $M_k = \mathbb{C}[x]/(x^k)$, $1 \leq k \leq n$ (Jordan blocks of size $\leq n$ with eigenvalue 0). Then

$$Q = \bullet \circlearrowleft \alpha, \qquad I = (\alpha^n).$$

General setting (Gabriel quiver). Q has one vertex v_S for each simple A-module S, and $\dim \operatorname{Ext}^1(T, S)$ arrows from v_S to v_T.

Example 25. In the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "\alpha_1"] \arrow[r, dashed] \arrow[r, bend right, "\alpha_k"'] & \underset{2}{\bullet} \end{tikzcd}$$

we have k exact sequences

$$0 \longrightarrow S(2) \hookrightarrow P_i(1) \twoheadrightarrow S(1) \longrightarrow 0,$$

namely, writing a representation as $V_1 \longrightarrow V_2$,

$$0 \longrightarrow \mathbb{C} \quad \hookrightarrow \quad \mathbb{C} \xrightarrow{\alpha_i \mapsto 1} \mathbb{C} \quad \twoheadrightarrow \quad \mathbb{C} \longrightarrow 0$$

where in $P_i(1)$ the arrow α_i acts as 1 and the other arrows act as 0.

Quantum groups

35. Hopf algebras and $U_q(\mathfrak{sl}_2)$

Quantum groups [Kas95], [CP94], [HK02].

Recall that $\mathfrak{sl}_2$ has generators e, f, h with relations

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

The *universal enveloping algebra* $U(\mathfrak{sl}_2)$ is the associative $\mathbb{C}$-algebra with generators e, f, h and relations

$$he - eh = 2e, \qquad hf - fh = -2f, \qquad ef - fe = h.$$

Poincaré–Birkhoff–Witt Theorem: $\left\{ e^a f^b h^c \;\middle|\; a, b, c \geq 0 \right\}$ forms a basis. (There is a similar construction for other Lie algebras.)

Definition 97.

a) A *bialgebra* (over $\mathbb{C}$) is an associative unital algebra H with multiplication $\mu\colon H \otimes H \to H$ and unit $\iota\colon \mathbb{C} \to H$, along with algebra homomorphisms $\Delta\colon H \to H \otimes H$ (*comultiplication*) and $\epsilon\colon H \to \mathbb{C}$ (*counit*) satisfying
   - *coassociativity*: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$;
   - *counit*: $(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$;
   - μ and ι are *coalgebra homomorphisms*: maps $\phi\colon C' \to C$ satisfying

   $$(\phi \otimes \phi) \circ \Delta_{C'} = \Delta_C \circ \phi, \qquad \epsilon_C \circ \phi = \epsilon_{C'}.$$

   (Without μ and ι, this defines a *coalgebra*.)

b) A *Hopf algebra* is a bialgebra along with an invertible linear map $S\colon H \to H$ (*antipode*) satisfying

   $$\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \iota \circ \epsilon = \mu \circ (\mathrm{id} \otimes S) \circ \Delta. \qquad (S \text{ is an algebra antiautomorphism})$$

c) Let τ be the flip map $a \otimes b \mapsto b \otimes a$. A bialgebra or Hopf algebra is *cocommutative* if $\tau \circ \Delta = \Delta$.

Examples:

a) $H = \mathbb{C}[G]$ (the group algebra), with

$$\Delta(g) = g \otimes g, \qquad \epsilon(g) = 1, \qquad S(g) = g^{-1}.$$

Check:

$$\text{coassoc.:} \quad (\Delta \otimes \mathrm{id}) \circ \Delta(g) = g \otimes g \otimes g = (\mathrm{id} \otimes \Delta) \circ \Delta(g);$$

antipode relation:

$$\begin{aligned} \mu \circ (S \otimes \mathrm{id}) \circ \Delta(g) &= \mu(S \otimes \mathrm{id})(g \otimes g) \\ &= \mu(g^{-1} \otimes g) = \underset{\in G}{1} = \iota \circ \epsilon. \end{aligned}$$

H is cocommutative: $\tau \circ \Delta(g) = g \otimes g = \Delta(g)$.

b) $H = U(\mathfrak{g})$, $\mathfrak{g}$ a Lie algebra, with

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \epsilon(x) = 0, \qquad S(x) = -x, \qquad x \in \mathfrak{g}.$$

Check:

$$\begin{aligned} \text{coassoc.:} \quad (\Delta \otimes \mathrm{id}) \circ \Delta(x) &= (\Delta \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) \\ &= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \\ &= (\mathrm{id} \otimes \Delta) \circ \Delta(x), \end{aligned}$$

$$\text{counit:} \quad (\epsilon \otimes \mathrm{id}) \circ \Delta(x) = (\epsilon \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) = x.$$

H is cocommutative: $\tau \circ \Delta(x) = 1 \otimes x + x \otimes 1 = \Delta(x)$.

If H is a Hopf algebra and U, V, W are H-modules, then

- H has a "trivial representation": $h \cdot \underset{\in \mathbb{C}}{z} := \epsilon(h) z$;
- V^* is an H-module via $(h \cdot \varphi)(v) := \varphi\big(S(x) \cdot v\big)$;
- $V \otimes W$ is an H-module via $h \cdot v \otimes w := \Delta(h)(v \otimes w)$;
- by coassociativity, $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$;
- if H is cocommutative, then $\tau \colon V \otimes W \to W \otimes V$ is an isomorphism.

What about a Hopf algebra that *is not* cocommutative?

Fix $q \in \mathbb{C}^\times$ or transcendental.

Definition 98. The *quantum group* $U_q(\mathfrak{sl}_2)$ is the associative $\mathbb{C}$-algebra generated by E, F, K, K^{-1} with relations

$$KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2} F,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

(Set $K = q^h$ and send $q \to 1$ to recover $U(\mathfrak{sl}_2)$.)

$U_q(\mathfrak{sl}_2)$ is a Hopf algebra:

$$\begin{array}{lll} \Delta(K) = K \otimes K & \epsilon(K) = 1 & S(K) = K^{-1} \\ \Delta(E) = 1 \otimes E + E \otimes K & \epsilon(E) = 0 & S(E) = -EK^{-1} \\ \Delta(F) = K^{-1} \otimes F + F \otimes 1 & \epsilon(F) = 0 & S(F) = -KF \end{array}$$

Not cocommutative!

Generically, the representation theory of $U_q(\mathfrak{sl}_2)$ is very similar to that of $U(\mathfrak{sl}_2)$ or $\mathfrak{sl}_2$.

Theorem 99. *Suppose that $q \in \mathbb{C}^\times$ is not a root of unity. The irreducible finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ are parametrized by nonnegative integers k. The irrep $V^{(k)}$ has a basis of K-eigenvectors $v_k, v_{k-2}, \ldots, v_{-k}$ such that*

$$\begin{aligned} Kv_j &= q^j v_j \\ Ev_{k-2\ell} &= [\ell]_q \, v_{k-2\ell+2} \\ Fv_{k-2\ell} &= [k-\ell]_q \, v_{k-2\ell-2} \end{aligned}$$

where

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = \underbrace{q^{n-1} + q^{n-3} + \cdots + q^{-(n-1)}}_{n \text{ terms}}.$$

Examples:

a) Standard representation ($k = 1$): $V^{(1)} = \mathbb{C}^2$,

$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}, \qquad E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

b) $k = 2$:

$$K \mapsto \begin{bmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{bmatrix}, \quad E \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & q+q^{-1} \\ 0 & 0 & 0 \end{bmatrix}, \quad F \mapsto \begin{bmatrix} 0 & 0 & 0 \\ q+q^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Exercise 13. Check that these matrices satisfy the relations of $U_q(\mathfrak{sl}_2)$.

36. R-matrices and quasitriangularity

Let $V = V_q^{(1)} = \mathbb{C}^2$ be the $U_q(\mathfrak{sl}_2)$-irrep with highest weight 1 (the standard representation). We have $V = \operatorname{span}\{v_+, v_-\}$ where

$$Kv_\pm = q^{\pm 1} v_\pm, \qquad \begin{array}{ll} Ev_+ = 0 & Fv_+ = v_- \\ Ev_- = v_+ & Fv_- = 0 \end{array}$$

$V \otimes V$ has basis $\{v_{++}, v_{+-}, v_{-+}, v_{--}\}$, where $v_{\pm\pm} = v_\pm \otimes v_\pm$.

Let $\widehat{R}\colon V \otimes V \to V \otimes V$ be the map

$$\begin{array}{c|cccc|c} & v_{++} & v_{+-} & v_{-+} & v_{--} & \\ \hline & q & 0 & 0 & 0 & v_{++} \\ \widehat{R} = & 0 & 0 & 1 & 0 & v_{+-} \\ & 0 & 1 & q-q^{-1} & 0 & v_{-+} \\ & 0 & 0 & 0 & q & v_{--} \end{array} \qquad \text{(if } q \to 1,\ \widehat{R} \to \tau\text{)}$$

Proposition 100. *$\widehat{R}$ is a $U_q(\mathfrak{sl}_2)$-module isomorphism. That is,*

$$\widehat{R} \circ \Delta(x) = \Delta(x) \circ \widehat{R} \qquad \textit{for all } x \in U_q(\mathfrak{sl}_2).$$

Proof. $\widehat{R}$ is bijective because it is nonsingular. Since $\Delta(K) = K \otimes K$ is symmetric, $\widehat{R}$ commutes with $\Delta(K)$ if and only if it preserves the weight spaces. These are:

$$\begin{array}{ll} \text{span}\,\{v_{++}\} & \text{weight } q^2 \\ \text{span}\,\{v_{+-}, v_{-+}\} & \text{weight } 1 \\ \text{span}\,\{v_{--}\} & \text{weight } q^{-2} \end{array}$$

This holds if $\widehat{R}$ has the block diagonal form

$$\left[\begin{array}{c|cc|c} * & & & \\ \hline & * & * & \\ & * & * & \\ \hline & & & * \end{array}\right],$$

which it does.

Since $\widehat{R}$ acts by a scalar on 1-dimensional weight spaces, we only need to check that it commutes with $\Delta(E)$ and $\Delta(F)$ on span $\{v_{+-}, v_{-+}\}$. We have

$$\begin{aligned} \Delta(E)\widehat{R}(v_{+-}) &= \Delta(E)(v_{-+}) = (1 \otimes E + E \otimes K)(v_{-+}) = 0 + q\, v_{++} \\ \widehat{R}\Delta(E)(v_{+-}) &= \widehat{R}(1 \otimes E + E \otimes K)(v_{+-}) = \widehat{R}(v_{++} + 0) = q\, v_{++} \end{aligned}$$

and

$$\begin{aligned} \Delta(E)\widehat{R}(v_{-+}) &= (1 \otimes E + E \otimes K)\big(v_{+-} + (q-q^{-1})v_{-+}\big) \\ &= v_{++} + 0 + 0 + q(q-q^{-1})v_{++} = q^2 v_{++} \\ \widehat{R}\Delta(E)(v_{-+}) &= \widehat{R}\big(0 + q\, v_{++}\big) = q^2 v_{++}, \end{aligned}$$

and similarly for $\Delta(F)$. $\square$

In fact, $\widehat{R}$ satisfies the *Yang–Baxter equation*:

$$\widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{12} = \widehat{R}_{23}\widehat{R}_{12}\widehat{R}_{23} \ \in \ \text{End}(V^{\otimes 3}),$$

where $\widehat{R}_{ij}$ acts as $\widehat{R}$ on the ith and jth tensor factors. More commonly, this is stated

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ \in \ \text{End}(V^{\otimes 3}), \qquad \text{where } R = \tau\widehat{R}.$$

In addition, by direct computation, we obtain the *quadratic relation*

$$\big(\widehat{R} - q\big)\big(\widehat{R} + q^{-1}\big) = 0.$$

Definition 101. A Hopf algebra H is *quasicocommutative* if there exists an invertible element $R \in H \otimes H$ such that

$$\tau\Delta(x) = R\,\Delta(x)\,R^{-1} \qquad \text{for all } x \in H.$$

H is *quasitriangular* if R further satisfies

$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \qquad (\mathrm{id} \otimes \Delta)R = R_{13}R_{12}.$$

This R is called the *universal R-matrix*, and satisfies the Yang–Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.

Theorem 102 (Drinfeld)**.** *Let $H = U_q(\mathfrak{sl}_2)$. The matrix*

$$R = q^{(H\otimes H)/2} \sum_{n\geq 0} \frac{(q-q^{-1})^n}{[n]_q!}\, q^{n(n-1)/2}\, E^n \otimes F^n \ \in\ H \mathbin{\widehat{\otimes}} H,$$

where $q^{(H\otimes H)/2}$ acts on the weight space $V_k \otimes V_\ell$ by $q^{k\ell/2}$ and $[n]_q! = [n]_q\,[n-1]_q \cdots [1]_q$, satisfies the conditions of Definition 101.

Similarly $U_q(\mathfrak{g})$ is "essentially quasitriangular" for any reductive Lie algebra $\mathfrak{g}$.

Similar to $U_q(\mathfrak{sl}_2)$, $H = U_q(\mathfrak{gl}_n)$ has a highest-weight representation V_q^λ which is a deformation of the $U(\mathfrak{gl}_n)$-representation V^λ.

Let $V = V_q^{(1)} = \mathbb{C}^n$, the "standard" $U_q(\mathfrak{gl}_n)$-representation. Then $U_q(\mathfrak{gl}_n)$ acts on $V^{\otimes k}$ via $\Delta^{(k)} = \Delta \circ \cdots \circ \Delta \colon H \to H^{\otimes k}$.

The braid group

$$B_k = \left\langle \sigma_1, \ldots, \sigma_{k-1} \;\middle|\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2 \right\rangle$$

also acts on $V^{\otimes k}$ via $\sigma_i \mapsto \widehat{R}_{i,i+1}$.

In fact $\widehat{R}$ also satisfies the quadratic relation $(\widehat{R}-q)(\widehat{R}+q^{-1}) = 0$, so we obtain a representation of the *Hecke algebra*

$$\mathcal{H}_k(q) = \left\langle T_1, \ldots, T_{k-1} \;\middle|\; T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ T_iT_j = T_jT_i,\ |i-j| \geq 2,\ (T_i-q)(T_i+q^{-1}) \right\rangle$$

via $T_i \mapsto \widehat{R}_{i,i+1}$.

Theorem 103 (Jimbo)**.** *The quantum group $U_q(\mathfrak{gl}_n)$ and the Hecke algebra $\mathcal{H}_k(q)$ are mutual centralizers inside $V^{\otimes k}$. We have the $U_q(\mathfrak{gl}_n) \otimes \mathcal{H}_k(q)$-module decomposition*

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda\vdash k \\ \ell(\lambda)\leq n}} V_q^\lambda \otimes S_q^\lambda,$$

where S_q^λ is the $\mathcal{H}_k(q)$-irrep deforming the Specht module S^λ.

37. Crystal bases

The references for this section are [HK02] and [BS17].

Two desiderata:

1) In a highest weight representation V^λ, the raising/lowering operators send weight spaces into weight spaces. We would like to extend this to the basis vectors themselves, by choosing a basis for each weight space V^λ_μ such that the raising/lowering operators send basis vectors to (scalar multiples of) basis vectors or 0.
   For $\mathfrak{gl}_n/\mathfrak{sl}_n$ there is a natural indexing set, since
$$\dim V^\lambda_\mu = k_{\lambda\mu} = \#\left\{\text{semistandard tableaux of shape } \lambda \text{ and content } \mu\right\}.$$
   For $\mathfrak{sl}_2$, this choice is possible since all weight spaces are 1-dimensional.
   For general $\mathfrak{g}$, it is not possible for most representations, e.g.
$$\begin{tikzcd} \bullet \arrow[dr, shift left, "f_i"'] & & \bullet \arrow[dl, shift left, "f_j"] \\ & \mathbb{C}^2 \arrow[ul, shift left, "e_i"'] \arrow[ur, shift left, "e_j"] & \end{tikzcd}$$
   usually will not lead to a compatible choice of basis for $\mathbb{C}^2$. Same issue for $U_q(\mathfrak{g})$ with generic q.

2) In a tensor product $V^{\otimes k}$ of copies of the standard representation, we would like the decomposition into irreps to partition the basis vectors.
   E.g. $U_q(\mathfrak{sl}_2)$, $V = V_q^{(1)} = \operatorname{span}\{v_+, v_-\}$. The irreducible decomposition is
$$V \otimes V \cong V_q^{(2)} \oplus V_q^{(0)},$$
   where
$$\begin{aligned} V_q^{(2)} &= \operatorname{span}\big(v_{++},\ v_{+-} + q\, v_{-+},\ v_{--}\big) \\ V_q^{(0)} &= \operatorname{span}\big(v_{-+} - q\, v_{+-}\big) \end{aligned}$$
   v_{+-} and v_{-+} lie in neither subrepresentation … unless we can set $q = 0$.

Theorem 104 (Lusztig, Kashiwara)**.** *For any reductive Lie algebra $\mathfrak{g}$, and for every highest weight representation V_q^λ of $U_q(\mathfrak{g})$, after taking an appropriate limit $q \to 0$, there exists a basis of the resulting representation satisfying 1). In many cases (and always in type A), it also satisfies 2).*

This is called the *crystal basis* for V^λ or V_q^λ. The *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i$ send basis vectors to basis vectors. The *crystal graph* has vertices the basis elements, and edges $v \xrightarrow{i} w$ when $\tilde{f}_i(v) = w$.

Let us work in the case $\mathfrak{g} = \mathfrak{sl}_n$. Set
$$\begin{aligned} V := V^{(1)} &= \operatorname{span}\left\{\text{semistandard tableaux with shape } \boxed{} \text{ and entry } \leq n\right\} \\ &= \operatorname{span}\left\{\boxed{1}, \boxed{2}, \dots, \boxed{n}\right\}. \end{aligned}$$

The crystal graph is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

$$\tilde{f}_i\left(\boxed{j}\right) = \begin{cases} \boxed{j{+}1}, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \qquad \tilde{e}_i\left(\boxed{j}\right) = \begin{cases} \boxed{j{-}1}, & \text{if } i = j-1, \\ 0, & \text{otherwise.} \end{cases}$$

In a tensor product $V^{\otimes k}$, $\tilde{f}_i(\boxed{a_1} \otimes \cdots \otimes \boxed{a_k})$ is given by:

- write) under each i, (under each $i+1$;
- change the rightmost unpaired i to an $i+1$.

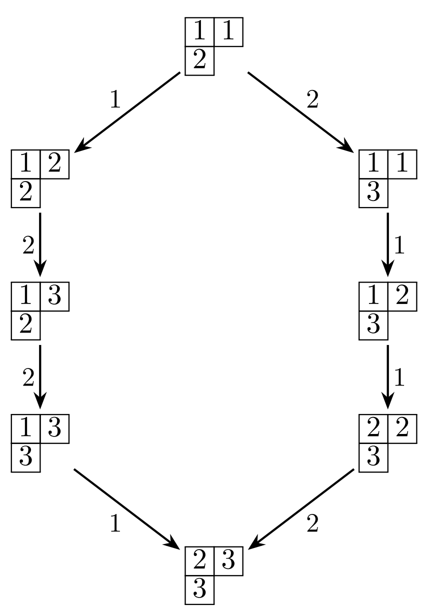
For example,

$$\begin{array}{rccccccccccccccccccccccc}
\tilde{f}_2(& 2 & \otimes & 1 & \otimes & 3 & \otimes & 3 & \otimes & 2 & \otimes & 1 & \otimes & 2 & \otimes & {\color{blue}2} & \otimes & 3 & \otimes & 1 & \otimes & 2 &) \\
 &) & & & & (& & (& &) & & & &) & & {\color{blue})} & & (& & & &) & \\
= & 2 & \otimes & 1 & \otimes & 3 & \otimes & 3 & \otimes & 2 & \otimes & 1 & \otimes & 2 & \otimes & {\color{red}3} & \otimes & 3 & \otimes & 1 & \otimes & 2 & \\
 &) & & & & (& & (& &) & & & &) & & {\color{red}(} & & (& & & &) &
\end{array}$$

Tableaux of shape λ fit into this framework by *row reading*, e.g.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \longmapsto \boxed{3} \otimes \boxed{1} \otimes \boxed{2},$$

and we obtain a crystal graph for any V^λ. For example, $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = (2, 1)$:



The tensor product rule allows us to decompose the tensor product of any two highest weight representations. E.g. $\mathfrak{g} = \mathfrak{sl}_2$, $V = V^{(2)}$, $W = V^{(3)}$:

$$V: \quad \begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \longrightarrow \begin{array}{|c|c|}\hline 2&2\\\hline\end{array}$$

$$W: \quad \begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}$$

$V \otimes W$:

$$\begin{tikzcd}
 & & 11 \otimes 112 & 11 \otimes 111 & \\
11222 \arrow[d] & 11122 \arrow[l] \arrow[d] & 11112 \arrow[l] \arrow[d] & 11111 \arrow[l] \arrow[d] & \\
12222 \arrow[d] & 12122 \arrow[l] \arrow[d] & 12112 \arrow[l] \arrow[d] & 12111 \arrow[l] \arrow[d] & 12 \otimes 111 \\
22222 & 22122 \arrow[l] & 22112 \arrow[l] & 22111 \arrow[l] &
\end{tikzcd}$$

$$V \otimes W \cong V^{(5)} \oplus V^{(3)} \oplus V^{(1)}.$$

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