Math 418, Spring 2025 – Practice Problems for Final Exam

8.2.2 *Prove that any two nonzero elements of a P.I.D. have a least common multiple*

9.4.11 *Prove that* $x^2 + y^2 - 1$ *is irreducible in* $\mathbb{Q}[x, y]$.

13.2.15 *A field* F *is said to be* formally real *if* -1 *is not expressible as a sum of squares in* F. *Let* F *be a formally real field, let* $f(x) \in F[x]$ *be an irreducible polynomial of odd degree and let* α *be a root of* $f(x)$. *Prove that* $F(\alpha)$ *is also formally real.*

13.5.8 *Prove that* $f(x)^p = f(x^p)$ *for any polynomial* $f(x) \in \mathbb{F}_p[x]$.

14.1.6 *Let* k *be a field. Show that the automorphisms of* $k[t]$ *that fix* k *are precisely the maps* $\phi(f(t)) = f(at + b)$, *for* $a, b \in k, a \neq 0$

14.2.23 *Let* K *be a Galois extension of* F *with cyclic Galois group of order* n *generated by* σ. *Suppose* $\alpha \in K$ *has* $N_{K/F}(\alpha) = 1$. *Prove that* α *is of the form* $\alpha = \frac{\beta}{\sigma\beta}$ *for some nonzero* $\beta \in K$.

15.2.8 *Suppose the prime ideal* P *contains the ideal* I. *Prove that* P *contains the radical of* I.

CLO-8.3.1

(a) *Show that* $I = (x^2y - x^3)$ *is a homogeneous ideal in* $k[x, y]$

(b) *Show that* $(f) \subseteq k[x_0, \dots, x_n]$ *is a homogeneous ideal if and only if* f *is a homogeneous polynomial*

Scheme Describe the scheme Spec $\mathbb{Z}$ to the level discussed in class. That is, say what the set of points is, what the functions are, how to evaluate a function at a point of Spec $\mathbb{Z}$, and what the sets of functions are on each of the sets $D(f)$.

G-C *Galois correspondence for the following polynomials* [Note: below, we'll assume over $\mathbb{Q}$. Over a finite field, the extension degree of the splitting field is just the degree of the largest irreducible factor, and the Galois group is the cyclic group of the appropriate size. Try to think about how this works and how the roots of each irreducible factor get permuted.]

(a) $x^9 - 1$

(b) $x^6 - 2$

(c) $x^3 + x + 1$

(d) $(x^2 - x - 1)(x^2 - 5)$