Math 418, Spring 2025 – Practice Problems 3

14.2.3 *Determine the Galois group of* $(x^2-2)(x^2-3)(x^2-5)$. *Determine all the subfields of the splitting field of this polynomial.*

14.2.10 *Determine the Galois group of the splitting field over* $\mathbb{Q}$ *of* x^8-3.

14.2.13 *Prove that if the Galois group of the splitting field of a cubic over* $\mathbb{Q}$ *is the cyclic group of order 3 then all the roots of the cubic are real.*

14.3.1 *Factor* x^8-x *into irreducibles in* $\mathbb{Z}[x]$ *and in* $\mathbb{F}_2[x]$.

14.4.4 *Let* $f(x)\in F[x]$ *be an irreducible polynomial of degree* n *over the field* F, *let* L *be the splitting field of* $f(x)$ *over* F *and let* α *be a root of* $f(x)$ *in* L. *If* K *is any Galois extension of* F, *show that the polynomial* $f(x)$ *splits into a product of* m *irreducible polynomials each of degree* d *over* K, *where* $d=[K(\alpha):K]=[(L\cap K)(\alpha):L\cap K]$ *and* $m=n/d=[F(\alpha)\cap K:F]$.

14.5.2 *Determine the subfields of* $Q(\zeta_8)$ *generated by the periods of* ζ_8 *and in particular show that not every subfield has such a period as primitive element.*

14.6.2a *Determine the Galois group of* x^3-x^2-4

14.6.3 *Prove for any* $a,b\in\mathbb{F}_{p^n}$ *that if* $f(x)=x^3+ax+b$ *is irreducible then* $-4a^3-27b^2$ *is a square in* $\mathbb{F}_{p^n}$

14.7.3 *Let* F *be a field of characteristic* $\neq 2$. *State and prove a necessary and sufficient condition on* $\alpha,\beta\in F$ *so that* $F(\sqrt{\alpha})=F(\sqrt{\beta})$. *Use this to determine whether* $\mathbb{Q}(\sqrt{1-\sqrt{2}})=\mathbb{Q}(i,\sqrt{2})$

A-G *Determine the following:*

(a) The radical of the ideal $(18)\subseteq\mathbb{Z}$

(b) The variety $V(I)\subseteq\mathbb{R}^2$ for the ideal $I=(x^2+y^2-1)\subseteq\mathbb{R}[x,y]$.