Announcements:

Final exam: Tues. 5/13 8:00am - 11:00am,
1047 Sidney Lu Mech. E. Bldg.
Exam will be cumulative
See policy email for full details

- TWO reference sheets allowed
- practice problems (sol'ns by the weekend)
- regrade requests will only span a couple days

Review session: Sunday 5/11 10:00am - 12:00pm
Altgeld 143
(come w/ questions)

Office hours: Friday and Monday 12:00pm - 1:00pm
or by email/appointment

ICES questionnaires: go.illinois.edu/ices-online

Final exam review

(See previous review topics)

Integral domains, poly. rings, irreducibility

Basic tools: irreducibility, field ext'ns, degrees, splitting fields, min'l polys., tower law

Constructibility: 4 classical problems, type of ext'ns allowed

Separability: derivative criterion, irreds. over char 0 or fin. field

Galois theory:

- Compute Galois gps. (both up to isom. class and via generators and rel'ns)
- Galois correspondence (draw diagrams etc.)
- Solvability by radicals
- Examples: cyclotomic ext'ns, finite fields, cubics, composite ext'ns

Algebraic geometry:

- Ideals, varieties, basic properties
- Radical ideals, Nullstellensatz (all forms)

Noetherian rings

Prime $\leftrightarrow$ irred., max'l $\leftrightarrow$ pt.

Coordinate ring

Projective space (all def'ns)

Homogeneous ideals, projective varieties

Projective Nullstellensatz

Specific examples

Schemes (only the small amount we covered last time; also see practice problem)

- for studying, look at lecture notes, homework/midterm problems, practice problems, textbook
- midterm length < final exam length < 2 · midterm length
- understand how topics mesh (e.g. ED/PID/UFD w/ alg. geom.)
- understand theory and examples

Example problems:

1) a) Prove that $V = \{(a, a^2, a^3) \mid a \in k\}$ is an irreducible affine variety.

Pf: $V = V(I)$ for $I = (x^2 - y, x^3 - z)$, so V is a variety. We show V is irred. by showing that I is prime. Can show this using the def'n of prime: if $f \cdot g \in I$, f or $g \in I$. Alternatively,

$$k[x,y,z]/_I \cong k[x]$$

$$x \longmapsto x$$
$$y \longmapsto x^2$$
$$z \longmapsto x^3$$

and since $k[x]$ is an int. domain, I is prime.

b) Prove that $W = \{[b^3 : ab^2 : a^2b : a^3] \mid a, b \in \mathbb{C}$ not both $0\}$ is an irred proj. variety.

Pf: $W = V(J)$ where $J = (xw - yz, xz - y^2, yw - z^2)$ (fill in the details). J is a homog. ideal, So W is a proj. variety. J is prime since

$$\left.\begin{aligned} k[x,y,z,w]/J &\cong k[s,t] \\ x &\longmapsto s^3 \\ z &\longmapsto s^2t \\ y &\longmapsto st^2 \\ w &\longmapsto t^3 \end{aligned}\right\} \text{ needs more details}$$

So W is irred. $\square$

2) Compute the Galois gp. / Galois corresp. for $f(x) = (x^3+x+1)(x^3+1)$ over $\mathbb{F}_2$

Soln: $x^3+1 = (x+1)(x^2-x+1)$ (over any field)

Over $\mathbb{F}_2$, x^3+x+1 and x^2-x+1 are irred. (no roots)

So $f(x) = (x+1)(x^2-x+1)(x^3+x+1) \in \mathbb{F}_2[x]$.

By D&F Prop 14.18 (or Prop 14.19)

$$\begin{aligned} Sp_{\mathbb{F}_2} f = \mathbb{F}_{2^n} \text{ where } n &= \operatorname{lcm}(\text{degrees of irred factors}) \\ &= \operatorname{lcm}(1,2,3) \\ &= 6 \end{aligned}$$

We have $G := \mathrm{Gal}(f) = \mathrm{Gal}(\mathbb{F}_{64}/\mathbb{F}_2) = \mathbb{Z}/6\mathbb{Z}$

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Z}/2\mathbb{Z} \arrow[dr, no head] & & \mathbb{Z}/3\mathbb{Z} \arrow[dl, no head] \\
 & G &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \mathbb{F}_{64} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{F}_8 \arrow[dr, no head] & & \mathbb{F}_4 \arrow[dl, no head] \\
 & \mathbb{F}_2 &
\end{tikzcd}$$

G abelian, hence everything normal / Galois

3) Prove that a quotient of a PID R by a prime ideal I is again a PID.

Pf: If $I = (0)$, then $R/I \cong R$ is a PID

If $I \neq (0)$, then I is maximal (D&F Prop 8.7), so R/I is a field, hence a PID.

$\square$

4) Let K/F be a nontriv. Galois ext'n of odd order, and let $\alpha \in K \setminus F$. Prove that

$$|\{\sigma \in \mathrm{Gal}(K/F) \mid \sigma(\alpha) \neq \alpha\}| > |\{\sigma \in \mathrm{Gal}(K/F) \mid \sigma(\alpha) = \alpha\}|$$

Pf: Since K/F is Galois, $\mathrm{Gal}(K/F(\alpha))$ is a proper subgp. of $\mathrm{Gal}(K/F)$. Since $[K:F]$ is odd, so is

$|\mathrm{Gal}(K/F)|$, so every proper subgp. has index ≥ 3.

Therefore, the subset of $\mathrm{Gal}(K/F)$ of automs. that fix α is $\leq \frac{1}{3}$ of the total. $\square$

b) Give a nontrivial ext'n of odd order s.t.

$$|\{\sigma \in \mathrm{Aut}(K/F) \mid \sigma(\alpha) \neq \alpha\}| \leq |\{\sigma \in \mathrm{Aut}(K/F) \mid \sigma(\alpha) = \alpha\}|$$

Sol'n: Let $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt[3]{2})$

$\mathrm{Aut}(K/F) = 1$, so id is the only elt., and this fixes $\sqrt[3]{2} \in K \setminus F$. $\square$