Recall:

Def: (complex) projective space is the set

$$\mathbb{P}^n(\mathbb{C}) = \{\text{lines thru. origin in } \mathbb{C}^{n+1}\}$$

$$= \{a = (a_0, \dots a_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}\} / (a \sim \lambda a,\ \lambda \in \mathbb{C})$$

$$= \{[a_0 : \cdots : a_n]\}$$

Cor: $\mathbb{P}^n(\mathbb{C}) = \underbrace{\mathbb{C}^n}_{\text{first coord. 1}} \cup \underbrace{\mathbb{P}^{n-1}(\mathbb{C})}_{\text{first coord. 0}}$

Remark: In class, we dealt w/ real proj. space instead since it's easier to visualize; the construction is analogous

Want to define projective varieties in $\mathbb{P}^n(\mathbb{C})$

Problem: Need

$$f(a_1, \dots, a_n) = 0 \Leftrightarrow f(\lambda a_1, \dots, \lambda a_n),$$

and not all functions satisfy this property

Fix:

Def: $f(x_0, \dots, x_n) \in \mathbb{C}^{n+1}$ is homogeneous of degree d if every term has degree d

If f homog. of degree d

$$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$$

If $\lambda \neq 0$, $f(\lambda a_0, \dots, \lambda a_n) = 0 \iff f(a_0, \dots, a_n) = 0$

Def: If $f \in \mathbb{C}[x_0, \dots, x_n]$ homog.,

$$V(f) := \{[a_0 : \cdots : a_n] \in \mathbb{P}^n(\mathbb{C}) \mid f(a_0, \dots, a_n) = 0\}$$

is the projective variety assoc. to f.

Note: no nonzero ideal consists of only homog. polys.

e.g. (x^2, y^2) contains $x^2 y + y^2$

Write

$$\mathbb{C}[x_0, \dots, x_n] = \bigoplus_{d=0}^{\infty} A_d$$

where $A_d = \{f \in \mathbb{C}[x_0, \dots, x_n] \mid f \text{ is homog of deg. } d\}$

Any $f \in \mathbb{C}[x_0, \dots, x_n]$ can be written uniquely as

$$f = f_0 + f_1 + \cdots, \quad f_d \in A_d$$

Def: An ideal $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ is homogeneous if

$$f \in I \Longrightarrow f_d \in I \ \forall d$$

Equivalently, I is homog. if it has a generating set consisting only of homog. polys.

Ex: $\mathbb{C}[x,y]$

a) $(x+y,\ x^2+y^2)$ is homogeneous

b) $(x+y,\ x+y+x^2+y^2)$ is homogeneous since it equals $(x+y,\ x^2+y^2)$

c) $(y-x^2)$ is not homog. since $y-x^2 \in (y-x^2)$, but y and x^2 are not.

Def: Let $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ be a homog. ideal. Then

$$V(I) = \{ a = [a_0 : \cdots a_n] \in \mathbb{P}^n(\mathbb{C}) \mid f(a) = 0 \ \forall f \in I \}$$

$$= V(f^{(1)}) \cap \cdots \cap V(f^{(k)})$$

if $f^{(i)}$ homog. and $I = (f^{(1)}, \dots, f^{(k)})$

These $V(I)$ are called <u>projective varieties</u>

Prop: $I(V) := \{ f \in \mathbb{C}[x_0, \dots, x_n] \mid f(a) = 0 \ \forall a \in V \}$ is a homog. ideal

Prop: If I homog., $\sqrt{I}$ is homog.

Projective Nullstellensatz: $\exists$ inc. reversing inv. bijections

$$\left\{\begin{matrix}\text{nonempty}\\ \text{projective}\\ \text{varieties}\end{matrix}\right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{\begin{matrix}\text{radical homog. ideals}\\ \text{properly cont. in } (x_0,\dots,x_n)\end{matrix}\right\}$$

For these varieties/ideals, $V(I(V)) = V$ and $I(V(I)) = \sqrt{I}$.

any ideal $\subsetneq (x_0,\dots,x_n)$

What about ϕ?

$$I(\phi) = \mathbb{C}[x_0,\dots,x_n] \quad \text{and} \quad V(\mathbb{C}[x_0,\dots,x_n]) = \phi$$

But also:

$$V((x_0,\dots,x_n)) = \{\text{pts. in } \mathbb{P}^n(\mathbb{C}) \text{ where } x_0 = \dots = x_n = 0\} = \phi$$

since $0 \in \mathbb{P}^n(\mathbb{C})$

So,

$$\begin{tikzcd}
 & \mathbb{C}[x_0,\dots,x_n] \arrow[dl, bend left, "V"] \\
\phi \arrow[ur, bend left, "I"] & \\
 & (x_0,\dots,x_n) \arrow[ul, "V"]
\end{tikzcd}$$

Furthermore

$$\left\{\begin{array}{c}\text{irred.}\\ \text{nonempty}\\ \text{projective}\\ \text{varieties}\end{array}\right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{\begin{array}{c}\text{prime homog. ideals}\\ \text{properly cont. in } (x_0, \ldots, x_n)\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{points}\\ a=[a_0:\cdots:a_n]\end{array}\right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{\begin{array}{c}\text{maximal ideals}\\ I(a) = \left(\frac{x_i}{a_i} - \frac{x_j}{a_j} \;\middle|\; 0 \le i,j \le n\right)\end{array}\right\}$$

Pf Sketch of proj. Nullstellensatz: Let

Let I be a homog. ideal properly cont. in $(x_0, \ldots, x_n)$,

and let $V = V(I)$.

Let $V' = \{a \in \mathbb{C}^{n+1} \mid f(a) = 0 \ \forall f \in I\}$

By the affine Nullstellensatz, $I(V') = \sqrt{I}$

We have

$$(a_0, \ldots, a_n) \in V' \setminus \{0\} \Leftrightarrow [a_0 : \cdots : a_n] \in V,$$

so $\sqrt{I} = I(V') \subseteq I(V)$

Conversely, if f homog., nonconstant, then $f(0)=0$, so

$$f \in I(V) \Rightarrow f(a)=0 \;\forall a \in V$$

homog.

$$\Rightarrow f(a)=0 \;\forall a \in V'$$

$$\Rightarrow f \in I(V) = \sqrt{I}.$$

Therefore, $I(V(I)) = I$ for all homog. ideals properly cont. in $(x_0, \ldots, x_n)$

The rest follows by similar arguments to the affine case.

$\square$