Announcements

HW10 updated with remaining problems (due Wed. 4/7)

HW grading is now caught up (HW8 & HW9 graded)

Final exam: Tues. 5/13 8:00am–11:00am,
1047 Sidney Lu Mech. E. Bldg. (lecture room, not the midterm room)
(email ASAP w/ any issues)

Exam will be cumulative

Schedule:

Today, Friday: projective space, projective varieties

Monday: topic TBD; see poll in email

Wednesday: review

We'll also have a review session closer to the exam

Midterm 3 graded

Median: 53/70

Mean: 52.6/70

Std. dev: 11.3

Q1: 70%

Q2: 74%

Q3: 86%

Q4: 80%

Gradelines:

A-/A: 54 to 70

B+/B/B-: 33 to $54-\varepsilon$

C+/C/C-: 14 to $33-\varepsilon$

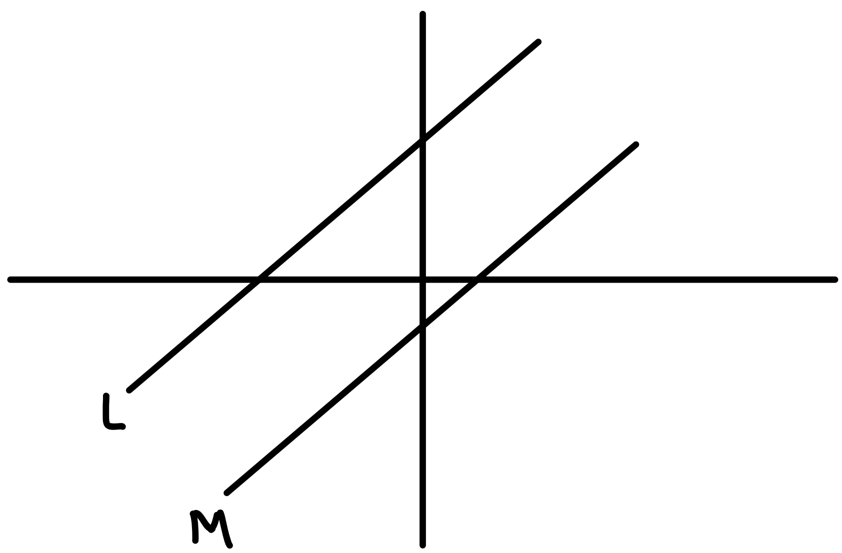
Sol'ns posted to website + gradeline spreadsheet updated

Projective space (see Cox, Little, O'Shea: Ideals, Varieties, and Algorithms, Ch 8.)

Motivation: recall

Bézout's Thm: The "usual" situation is that two poly. in $\mathbb{C}[x,y]$ of degrees m and n have $m \cdot n$ intersection points in $\mathbb{C}$

But what about parallel lines?



$(\deg L)(\deg M) = 1 \cdot 1 = 1$, but L and M don't intersect

Fix: add pts. "at ∞" where parallel lines meet

Consider equiv. classes of parallel lines

Def (version 1): The (complex) projective plane is the set

$$\widetilde{\mathbb{P}^2(\mathbb{C})} = \mathbb{C}^2 \cup \underbrace{\{\text{one pt "at } \infty\text{" for each equiv. class of parallel lines}\}}_{H_\infty}$$

for now, to distinguish from def. 2

Works, but kind of a weird defn

For a nicer one, let's define homogenous coords. in $\mathbb{C}^3$

We say that $\underbrace{(a_0, a_1, a_2)}_{\in \mathbb{C}^3} \sim (b_0, b_1, b_2)$

if $(b_0, b_1, b_2) = (\lambda a_0, \lambda a_1, \lambda a_2)$ for some $\lambda \in \mathbb{C} \setminus \{0\}$

i.e. if all the ratios are the same: $\frac{a_0}{a_1} = \frac{b_0}{b_1}$, $\frac{a_0}{a_2} = \frac{b_0}{b_2}$, $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

i.e. if $a, b \neq 0$, $a \sim b \iff a$ and b are on the same line thru. origin in $\mathbb{C}^3$

Denote equiv. classes $[a_0 : a_1 : a_2]$

Def (version 2): The complex proj. plane is the set of equivalence classes

$$\mathbb{P}^2(\mathbb{C}) = (\mathbb{C}^3 \setminus \{0\}) / \sim$$

i.e. the set of 1D subspaces of $\mathbb{C}^3$

Prop: There is a (nice) bijection

$$\underset{\text{def } 2}{\mathbb{P}^2(\mathbb{C})} \longrightarrow \underset{\text{def } 1}{\widetilde{\mathbb{P}^2(\mathbb{C})}}$$

$$\text{Pf: } \mathbb{P}^2(\mathbb{C}) = \underbrace{\{[1:x:y] \mid x, y \in \mathbb{C}\}}_{S_1} \cup \overbrace{\{[0:1:y] \mid y \in \mathbb{C}\}}^{S_2} \cup \underbrace{\{[0:0:1]\}}_{S_3}$$

$[1:x:y] \mapsto (x,y)$ is a bij. $S_1 \to \mathbb{C}^2$

Let $a_m \in H_\infty$, $m \in \mathbb{C} \cup \{\infty\}$ be the equiv. class of lines in $\mathbb{C}^2$ of slope m

Then
$$[0:1:m] \longmapsto a_m$$
$$[0:0:1] \longmapsto a_\infty$$

gives a bijection $S_2 \cup S_3 \longrightarrow H_\infty$

$\square$

Def: (complex) projective space is the set

$$\mathbb{P}^n(\mathbb{C}) = \{\text{lines thru. origin in } \mathbb{C}^{n+1}\}$$

$$= \{a = (a_0, \dots a_{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}\} \big/ (a \sim \lambda a,\ \lambda \in \mathbb{C})$$

$$= \{[a_0 : \cdots : a_n]\}$$

Cor: $\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \cup \mathbb{P}^{n-1}(\mathbb{C})$

Pf: Use the maps from the previous prop:

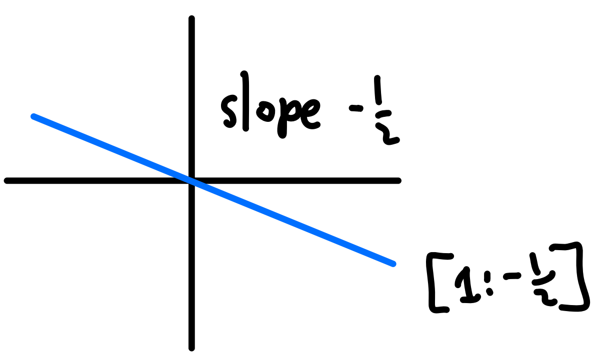
$$[1 : a_1 : \cdots : a_n] \longmapsto (a_1, \dots, a_n) \in \mathbb{C}^n$$

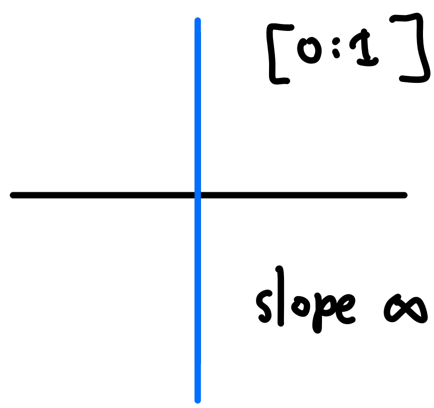
$$[0 : a_1 : \cdots : a_n] \longmapsto \underbrace{[a_1 : \cdots : a_n]}_{\text{not all } 0} \in \mathbb{P}^{n-1}(\mathbb{C})$$

$\square$

Ex: $\mathbb{P}^1(\mathbb{C}) = \{\text{lines in } \mathbb{C}^2\} = \{[x : y]\}$

$$= \{[1 : m] \mid m \in \mathbb{C}\} \cup \{[0 : 1]\}$$





Also called the Riemann sphere

Want to define projective varieties in $\mathbb{P}^n(\mathbb{C})$

Let $f(x,y,z) = xy - z$

Then $f(1,1,1) = 0$

$f(2,2,2) = 2$

So what does $f([1:1:1])$ mean?

Problem: when we scaled the variables, we doubled z but quadrupled xy

Fix:

Def: $f(x_0,..,x_n) \in \mathbb{C}^{n+1}$ is homogeneous of degree d if every term has degree d

If f homog. of degree d

$$f(\lambda a_0,\dots,\lambda a_n) = \lambda^d f(a_0,..,a_n)$$

If $\lambda \neq 0$, $f(\lambda a_0,\dots,\lambda a_n) = 0 \iff f(a_0,..,a_n) = 0$

Def: If $f \in \mathbb{C}[x_0,..,x_n]$ homog.,

$$V(f) := \{[a_0:\dots:a_n] \in \mathbb{P}^n(\mathbb{C}) \mid f(a_0,\dots,a_n) = 0\}$$

is the projective variety assoc. to f.

Next time: $V(I)$ for "homog. ideal" I