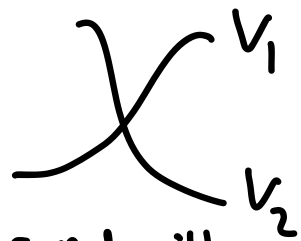
Announcements

First part of HW10 posted (due Wed. 5/7)
Rest will be posted next week

Recall: A variety V is irreducible if whenever $V = V_1 \cup V_2$ for varieties V_1 and V_2, $V = V_1$ or $V = V_2$.

Prop: V irred $\Leftrightarrow$ $I := I(V)$ prime



a reducible variety

Pf: $\Rightarrow$) Let $f_1 f_2 \in I$

Let $V_i = V \cap V(f_i) = V(I + (f_i))$ $\quad (i = 1,2)$

$= \{a \in V \text{ s.t. } f_i(a) = 0\}$

Let $a \in V$. Then $f_1(a) \cdot f_2(a) = f_1 f_2(a) = 0$, so $f_1(a) = 0$ or $f_2(a) = 0$, and so $V = V_1 \cup V_2$.

Since V irred, $V = V_j$ for $j = 1$ or 2, so $f_j(a) = 0$ for all $a \in V$, which means that $f_j \in I$, so I is prime.

$\Leftarrow$) Let $V = V_1 \cup V_2$, and assume $V_1 \subsetneq V$.

This means that $I(V) \subsetneq I(V_1)$ since otherwise

$V = V(I(V)) = V(I(V_1)) = V_1$.

Let $f_1 \in I(V_1) \setminus I(V)$, $f_2 \in I(V_2)$.

Then $f_1 f_2 \in I(V)$ since one of f_1, f_2 is 0 on every point in V.

Since $I(V)$ is prime, must have $f_2 \in I$ (can't have $f_1 \in I$), so $I(V_2) \subseteq I(V)$, so $V_2 \subseteq V \subseteq V_2$, so $V = V_2$ and V irred. $\square$

Prop: Any variety $V \subseteq k^n$ is a finite union of irred. varieties.

Def: A ring R is <u>Noetherian</u> if every strictly increasing chain of ideals is finite i.e. if

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

then $\exists m$ s.t. $I_k = I_m \ \forall k \geq m$

(sometimes called the ascending chain condition)

Hilbert's Basis Thm: $k[x_1,\ldots,x_n]$ is Noetherian

(Pf: D&F Section 9.6, Cor 9.22, uses "leading coeffs.")

Pf of prop: Suppose otherwise. Since V red.,

$$V = V_1 \cup W_1$$

varieties
$V_1, W_1 \subsetneq V$

One of V_1, W_1 must be reducible, say $V_1 = V_2 \cup W_2$, $V_2, W_2 \subsetneq V_1$. Continuing in this manner, we have

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots$$

and letting $I_i = I(V_i)$, we get

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$$

since $V(I_i) = V_i \supsetneq V_{i+1} = V(I_{i+1})$

Since $k[x_1,\ldots,x_n]$ is Noetherian, this is impossible. $\square$

What about maximal ideals?

max'l ideals $\subseteq$ prime ideals $\leftrightarrow$ irred. varieties

For $a \in k^n$, let $I(a) = \{f \in k[x_1, \dots, x_n] \mid f(a) = 0\} = I(\{a\})$

Lemma:

a) $I(a) = (x_1 - a_1, \dots, x_n - a_n)$

b) $I(a)$ is maximal

Pf: b) $I(a) = \ker(f \mapsto f(a))$, so

$$k[x_1, \dots, x_n]/J \cong \operatorname{im}(f \mapsto f(a)) = k,$$

a field, so $J = I(a)$ is max'l.

a) Let $J := (x_1 - a_1, \dots, x_n - a_n) \subseteq I(a)$. Suppose that $J \subsetneq I(a)$, and let $f \in I(a) \setminus J$ have smallest degree. f can't be constant, so if $c x_1^{e_1} \cdots x_n^{e_n}$ is a monomial of top degree, then $e_i > 0$ for some i. Then

$$f - (x_i - a_i) c x_1^{e_1} \cdots x_i^{e_i - 1} \cdots x_n^{e_n} \in I(a) \setminus J,$$

and $c x_1^{e_1} \cdots x_n^{e_n}$ has been replaced by the smaller-degree monomial $c a_i x_1^{e_1} \cdots x_i^{e_i - 1} \cdots x_n^{e_n}$. Doing this for every top-degree monomial of f we get an elt of $I(a) \setminus J$ with smaller top degree, a contradiction. $\square$