Announcements

Midterm 3: Wed 4/23, 7:00-8:30pm, Sidney Lu 1043

- Covers through Friday (start of algebraic geometry)
- Practice problem sol'ns posted
- Wednesday class will be review
- Office hour Wed. after class (+ usual prob. session)

Recall:

Unless otherwise stated, let k be an alg. closed field

Def: An (affine) <u>algebraic variety</u> (or <u>algebraic set</u>) is a subset $V \subseteq k^n$ of the form

$$V = V(I) := \{a \in k^n \mid f(a) = 0 \ \forall f \in V\}$$

for some subset/ideal $I \subseteq k[x_1, .., x_n]$

Def: V: alg. variety. Then set

$$I(V) = \{f \in k[x_1, .., x_n] \mid f(a) = 0 \ \forall a \in V\}$$

Radical of I:

$$\sqrt{I} = \{r \in k[x_1, .., x_n] \mid r^n \in I \text{ for some } n \geq 0\}$$

Prop: I, J: ideals

a) $I \subseteq J \Rightarrow V(I) \supseteq V(J)$

b) $V(I) \cap V(J) = V(I \cup J) = V(I+J)$

c) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$

d) $V(0) = k^n$ and $V(\langle 1 \rangle) = \phi$

Prop: U, V: varieties

a) $U \subseteq V \Rightarrow I(U) \supseteq I(V)$

b) $I(U \cup V) = I(U) \cap I(V)$

c) $I(U \cap V) \supseteq I(U) + I(V)$

Prop:

a) $V = V(I(V))$

b) $I \subseteq I(V(I))$

Hilbert's Nullstellensatz (weak form, first version):

Let $f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$

Then the system of equations

$$f_1(x_1, \dots, x_n) = \dots = f_m(x_1, \dots, x_n) = 0$$

has <u>no</u> solution in $\mathbb{C}^n$ if and only if

$\exists g_1, \dots, g_m \in \mathbb{C}[x_1, \dots, x_n]$ s.t. $f_1 g_1 + \dots + f_m g_m = 1 \in \mathbb{C}[x_1, \dots x_n]$

Hilbert's Nullstellensatz (strong form): $I(V(I)) = \sqrt{I}$.

Moreover, we have inverse bijections

$$\begin{tikzcd} \text{alg. varieties } V \subseteq k^n \arrow[r, shift left, "I"] & \text{radical ideals } I \subseteq k[x_1, \ldots, x_n] \arrow[l, shift left, "V"] \end{tikzcd}$$

Pf of easy direction: If $f \in \sqrt{I}$ then $f^n \in I$ for some n. If $a \in V(I)$, then

$0 = f^n(a) = (f(a))^n$, so $f(a) = 0$ since

$k[x_1, \ldots, x_n]$ is an int. domain, so $\sqrt{I} \subseteq I(V(I))$. $\square$

Cor: Hilbert's Nullstellensatz (weak form, second version)

Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then $V(I) = \emptyset$ if and only if $1 \in I$ (and so $I = k[x_1, \ldots, x_n]$)

Pf: By the strong form,

$$\sqrt{I} = I(V(I)) = I(\emptyset) = k[x_1, \ldots, x_n],$$

So $1 \in \sqrt{I}$. This means that $1^n \in I$ for some n,

so $1 = 1^n \in I$ $\square$

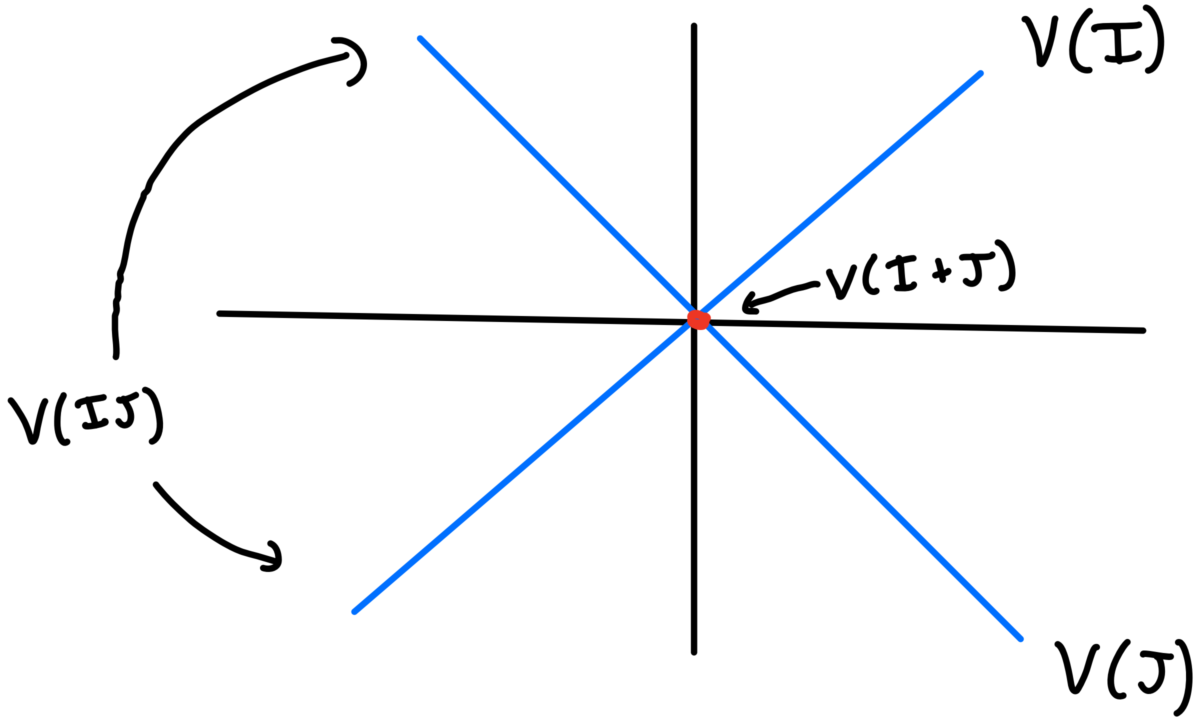
(in practice, the weak form is used to prove the strong form)

Examples:

a) $k = \mathbb{C}$ (or $\mathbb{R}$), $n = 2$

$I = (x-y)$, $\quad J = (x+y)$ $\quad I+J = (x,y)$

$I \cap J = IJ = ((x-y)(x+y))$



$$I(V(J)) = \{ f \in \mathbb{C}[x,y] \mid f(x,-x) = 0 \ \forall x \}$$

If $(x+y) \mid f(x,y)$, (recall: $k[x_1,\ldots,x_n]$ is a UFD)

then $f(x,-x)=0$

So $J \subseteq I(V(J))$. Can this containment be strict?

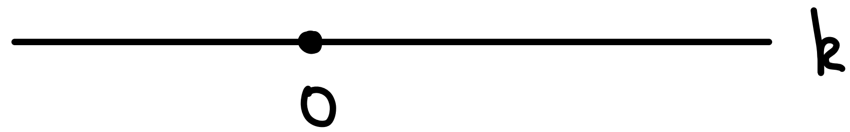
Yes, but in this case $I(V(J)) = J$

$$I(V(I+J)) = \{ f \in k[x,y] \mid f(0,0)=0 \}$$

= all functions w/out a constant term

$= (x,y) = I+J$

b) $n=1$ $\quad I = (x^2) \subseteq k[x]$



$V(I) = 0$, but $I(V(I)) = (x) \supseteq I$

$= \sqrt{I}$

[Aside: how would we distinguish (x^2) from (x)?

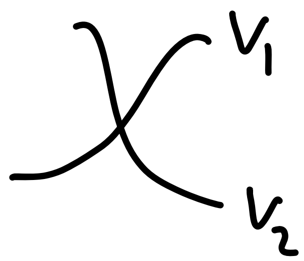
Ans: replace varieties with <u>schemes</u>]

Prime ideals are radical since in a prime ideal I,

$ab \in I \Rightarrow a \in I$ or $b \in I$, so $a^n \in I \Rightarrow a \in I$

Def: A variety V is irreducible if whenever $V = V_1 \cup V_2$ for varieties V_1 and V_2, $V = V_1$ or $V = V_2$.

Prop: V irred $\Leftrightarrow$ $I := I(V)$ prime



Pf: $\Rightarrow$) Let $f_1 f_2 \in I$

Let $V_i = V \cap V(f_i) = V(I + (f_i))$ $\quad (i = 1,2)$

$= \{a \in V \text{ s.t. } f_i(a) = 0\}$

Let $a \in V$. Then $f_1(a) \cdot f_2(a) = f_1 f_2(a) = 0$, so $f_1(a) = 0$ or $f_2(a) = 0$, and so $V = V_1 \cup V_2$.

Since V irred, $V = V_j$ for $j = 1$ or 2, so $f_j(a) = 0$ for all $a \in V$, which means that $f_j \in I$, so I is prime.

$\Leftarrow$) Let $V = V_1 \cup V_2$, and assume $V_1 \subsetneq V$.

This means that $I(V) \subsetneq I(V_1)$ since otherwise

$V = V(I(V)) = V(I(V_1)) = V_1$.

Let $f_1 \in I(V_1) \setminus I(V)$, $f_2 \in I(V_2)$.

Then $f_1 f_2 \in I(V)$ since one of f_1, f_2 is 0 on every point in V.

Since $I(V)$ is prime, must have $f_2 \in I$ (can't have $f_1 \in I$), so $I(V_2) \subseteq I(V)$, so $V_2 \subseteq V \subseteq V_2$, so $V = V_2$ and V irred. $\square$

Prop: Any variety $V \subseteq k^n$ is a finite union of irred. varieties.

Pf: Friday