Cubic formula

Thm (Cardono, 1545): The cubic eqn. is solvable by radicals.

$$f(x) = x^3 + ax^2 + bx + C$$

$$g(y) = f\left(x + \tfrac{a}{3}\right) = y^3 + py + q \qquad \text{"depressed cubic"}$$

$$p = \tfrac{1}{3}(3b - a^2) \qquad q = \tfrac{1}{27}(2a^3 - 9ab + 27c)$$

Let $\zeta = \zeta_3 \qquad \zeta^2 + \zeta + 1 = 0$

Let $g(y)$ have roots α, β, γ

Have $e_1 = \alpha + \beta + \gamma = -y^2\text{-coeff} = 0$

$e_2 = p \quad e_3 = -q$

Key idea: coeffs. are the sym. funs. in the roots, so find expressions for the roots themselves using these sym. funs.

$$0 = \alpha + \beta + \gamma$$

$$\left.\begin{aligned} \theta_1 &:= \alpha + \zeta\beta + \zeta^2\gamma \\ \theta_2 &:= \alpha + \zeta^2\beta + \zeta\gamma \end{aligned}\right\} \text{"Lagrange resolvents"}$$

$$\theta_1 + \theta_2 = 3\alpha$$

$$\rho^2 \theta_1 + \rho\theta_2 = 3\beta$$

$$\rho\theta_1 + \rho^2\theta_2 = 3\gamma$$

$$\sqrt{D} = (\alpha-\beta)(\alpha-\gamma)(\beta-\gamma) = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha - \alpha\beta^2 - \beta\gamma^2 - \gamma\alpha^2$$

So if $S = \alpha^2\beta + \beta^2\gamma + \gamma^2\alpha + \alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2$, then

$$\theta_1^3 = \alpha^3 + \beta^3 + \gamma^3 + 3\rho(\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha)$$
$$+ 3\rho^2(\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2) + 6\alpha\beta\gamma$$

$$= \alpha^3 + \beta^3 + \gamma^3 + \frac{3}{2}\rho(S + \sqrt{D}) + \frac{3}{2}\rho^2(S - \sqrt{D}) + 6\alpha\beta\gamma$$

Can show (given that $\alpha+\beta+\gamma=0$)

$$\alpha^3 + \beta^3 + \gamma^3 = -3q, \quad S = 3q$$

So

$$\theta_1^3 = -3q + \frac{3}{2}\rho(3q + \sqrt{D} + \frac{3}{2}\rho^2(3q - \sqrt{D}) - 6q$$

$$= -\frac{27}{2}q + \frac{3}{2}\sqrt{-3D} \qquad \left(\text{since } \begin{array}{l} \rho + \rho^2 = -1 \\ \rho - \rho^2 = \sqrt{-3} \end{array}\right)$$

Similarly,

$$\Theta_2^3 = -\frac{27}{2}q - \frac{3}{2}\sqrt{-3D}$$

(need $\Theta_1\Theta_2 = -3p$)

Choose

$$A = \sqrt[3]{-\frac{27}{2}q + \frac{3}{2}\sqrt{-3D}} \qquad B = \sqrt[3]{-\frac{27}{2}q - \frac{3}{2}\sqrt{-3D}}$$

s.t. $AB = -3p$. Then,

$$\alpha = \frac{A+B}{3} \qquad \beta = \frac{\rho^2 A + \rho B}{3} \qquad \gamma = \frac{\rho A + \rho^2 B}{3}$$

(Quartic formula follows from this and the resolvent cubic)

Solvability by radicals

Def: $f(x) \in F[x]$ is <u>solvable by radicals</u> if $\exists$

$$F = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_s = Sp_F f$$

where $K_{i+1} = K_i(\alpha_i)$ w/ α_i a root of $x^{n_i} - a_i$

Assume char $F = 0$ $\left(\begin{array}{l}\text{or just let it not divide anything} \\ \text{we don't want it to}\end{array}\right)$

Thm (ancients, Cardano, Ferrari): All deg ≤ 4 polys are solvable

Thm (Abel-Ruffini): There is no general formula by radicals for $f_{gen}^{(n)}$, $n \geq 5$.

Thm (Galois):

a) $f(x)$ is solvable by radicals $\iff$ Gal f is a "solvable gp"

b) $\exists$ a degree 5 poly. which is not solvable by radicals.

Def: A finite gp. G is <u>solvable</u> (UK: "soluable") if

$$\{1\} = G_s \triangleleft G_{s-1} \triangleleft \cdots \triangleleft G_0 = G$$

where G_i/G_{i+1} is cyclic.

Examples:

- abelian gps.
- dihedral gps.

$$1 \trianglelefteq C_n \underbrace{\trianglelefteq}_{C_2} D_{2n}$$

- p-gps. ($|G| = p^k$)
- S_4: $1 \underbrace{\trianglelefteq}_{V_4} V_4 \underbrace{\trianglelefteq}_{C_2} A_4 \underbrace{\trianglelefteq}_{C_2} S_4$

Non-examples:

- S_n or $\underbrace{A_n \text{ for } n \geq 5}_{\text{no normal subgps! i.e. "simple"}}$ (D&F Thm 4.24)

- Other finite simple gps. (e.g. the monster)

Cor: If $n = 5$, $K = Sp_F f$,

$Gal(K/F) = S_n$ or $A_n \Rightarrow f$ is not solvable by radicals

So Galois $\Rightarrow$ Abel-Ruffini

Prop:

a) If $H \leq G$, then G solvable $\Rightarrow$ H solvable

b) If $H \triangleleft G$, then H solvable, G/H solvable $\Rightarrow$ G solvable

Pf:

a) Let $\{1\} = G_s \triangleleft G_{s-1} \triangleleft \cdots \triangleleft G_0 = G$

where G_i/G_{i+1} is cyclic, and let $H_i = H \cap G_i$

Then $H_{i+1} \triangleleft H_i$ and H_{i+1}/H_i is isom to a subgp. of G_{i+1}/G_i, so is cyclic.

b) $1 = H_s \triangleleft H_{s-1} \triangleleft \cdots \triangleleft H_0 = H$

$1 = J_r \triangleleft J_{r-1} \triangleleft \cdots \triangleleft J_r = G/H$

If $\pi : G \to G/H$, then

$$1 = H_s \triangleleft \cdots \triangleleft H_0 = \pi^{-1}(J_r) \triangleleft \pi^{-1}(J_{r-1}) \triangleleft \cdots \triangleleft \pi^{-1}(J_1) = G$$

cyclic

□