Announcement

Wednesday's class will be observed

Symmetric functions and the discriminant

Let $f(x) \in F[x]$, $K = Sp_F f$

Def: The Galois gp. of $f(x)$ is $Gal(f) := Gal(K/F)$

We want to understand $Gal(f)$ for different polys.

Thm (Abel, Ruffini): The degree-5 poly. is not solvable by radicals

We know: If $\deg f = n$, $Gal(f) \leq S_n$

Generic version:

$$K = F(\underbrace{x_1, \ldots, x_n}_{\substack{\text{think of these} \\ \text{as "roots" of} \\ \text{a "generic poly"}}}) = \begin{matrix}\text{field of fractions} \\ \text{of } F[x_1, \ldots, x_n]\end{matrix} = \left\{ \frac{3x_1^2 x^3 - 5x_2}{1 + x_4 + x_1^4 x_2}, \ldots \right\}$$

We have $S_n \leq Aut(K/F)$ (permute the x_i's)

Set $L = \text{Fix}\, S_n$, and we have $\text{Gal}(K/L) = S_n$

field of symmetric functions

Example elts:

- $f \in F$
- $e_1 = e_1(x_1, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$
- $e_2 = \sum_{i<j} x_i x_j = x_1x_2 + x_1x_3 + x_2x_3 + \cdots$

  $\vdots$
- $e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$

elementary sym. funs.
(D&F call them s_k)

Fun. Thm. of Sym. Funs: $L = F(e_1, \ldots, e_n)$

Pf: Let $L' = F(e_1, \ldots e_n)$. Then $L' \subseteq L$ and $[K:L] = |S_n| = n!$, so we just need to show that $[K:L'] \leq n!$. This follows since K is the splitting field of the following deg. n poly in $L'[x]$:

$$f_{gen}^{(n)}(x) = \prod_i (x - x_i)$$

$$= x^n - (x_1 + \dots + x_n)x^{n-1} + \dots + (-1)^n x_1 \cdots x_n$$

$$= x^n - e_1 x^{n-1} + \dots + (-1)^n e_n \qquad \square$$

Def: The <u>discriminant</u> of $f(x) \in F[x]$ is

$$D = \prod_{i<j} (\alpha_i - \alpha_j)^2$$

where α_i are the roots of F in $K := Sp_F(f)$

Prop: $D = 0 \iff f$ is inseparable.

Prop: $D \in F$

Pf: D is sym. in the α_i, so

$$D \in F(\underbrace{e_1(\alpha_1, \dots, \alpha_n), \dots, e_n(\alpha_1, \dots, \alpha_n)}_{\text{coeffs. of } f}) = F \qquad \square$$

E.g.:

a) $f = f^{(2)}_{gen}(x) = (x - x_1)(x - x_2)$

$$D = (x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2$$
$$= (x_1 + x_2)^2 - 4x_1x_2$$
$$= e_1^2 - 4e_2$$

So if

$$f(x) = x^2 + \underbrace{b}_{-e_1}x + \underbrace{c}_{e_2}, \text{ then } D = b^2 - 4c \quad (!)$$

b) If $f(x) = x^3 + ax^2 + bx + c$,

$$D = a^2b^2 - 4b^3 - 4a^3c - 27c^2 + 18abc$$

Take a sqrt:

$$\sqrt{D} = \prod_{i<j} (\alpha_i - \alpha_j)$$

$$\begin{array}{c} K = F(\alpha_1, \ldots, \alpha_n) \\ | \\ F(\sqrt{D}) \\ | \\ F = F(D) \end{array}$$

Assume char $F \neq 2$

If $G := \mathrm{Gal}(K/F) = S_n$

then $\exists \sigma \in D$ w/ $\sigma(\sqrt{D}) = -\sqrt{D}$. Thus, $\sqrt{D} \notin F$

e.g. $\sigma = (12)$

Recall: $A_n = \left\{ \begin{matrix} \text{even perms.} \\ \text{of } 1, \ldots, n \end{matrix} \right\} \leq S_n$ ← index 2

Prop: $G \leq A_n \iff \sqrt{D} \in F$

Pf: $\sigma(\sqrt{D}) = \sqrt{D} \iff \sigma$ is even, so

$$G \leq A_n \iff \sigma(\sqrt{D}) = \sqrt{D} \quad \forall \sigma \in G$$

$$\iff \sqrt{D} \in \mathrm{Fix}\, G = F \qquad \square$$

Next time: find Galois gps of small degree polys.