Announcements

No class or office hour this Friday

Today I will stick around for ~20 minutes after class

HW5 posted (due Wed. 3/5 @ 9:00 am)

Recall: A splitting field $Sp(f)$ of a poly. $f \in F[x]$ is an extension field over which f splits completely, and which is minimal w.r.t. this property

Last time: existence

Next: uniqueness

Thm: Let $\varphi: F \xrightarrow{\sim} F'$ be an isom. of fields. Let $f(x) \in F[x]$, and $f'(x)$ be the image of f in $F'[x]$ under φ (mapping x to itself)

a) Suppose f is irred. Let α be a root of f, β be a root of f'. Then $\exists\ F(\alpha) \xrightarrow{\sim} F'(\beta)$ sending $F \xrightarrow{\varphi} F'$, $\alpha \mapsto \beta$

b) Let K be a splitting field for f over F

K' be a splitting field for f' over F'

Then $\exists\ K \xrightarrow{\sim} K'$ sending $F \xrightarrow{\varphi} F'$

Pf: a) $F(\alpha) \cong F[x]/_{(f)} \cong F'[x]/_{(f')} \cong F'(\beta)$

b) Induction. Choose a root $\alpha \in K$ of some irred. factor p of f and a root $\beta \in K'$ of $p' := \varphi(p)$.

By part a), $F(\alpha) \cong F'(\beta)$, so let $E := F(\alpha)$, $E' := F'(\beta)$.

Now if $g = \frac{f}{x-\alpha}$, $g' = \frac{f'}{x-\beta}$, we have the same situation as b) but w/ g, g', E, E' replacing f, f', F, F'.

By the inductive hypothesis, $\exists\ K \xrightarrow{\sim} K$

sending $E \xrightarrow{\sim} E'$

sending $F \xrightarrow{\sim} F'$. $\square$

Cor: $Sp_F f$ is unique up to isom. $\square$

Ok, we can get one poly. to split. What about <u>all</u> polys.?

Def: We'll use this notation

a) $\overline{F}$ is an algebraic closure of F if $\overline{F}/F$ is alg. and every $f(x) \in F[x]$ splits completely in $\overline{F}[x]$

(equivalently, every nonconstant $f(x) \in F[x]$ has a root in $\overline{F}$)

b) K is alg. closed if $\overline{K} = K$

Prop: Alg. closure $\Longrightarrow$ alg. closed
(i.e. If $K = \overline{F}$, $K = \overline{K}$)

Pf:
$$\underbrace{F \underset{\text{alg.}}{\subseteq} K = \overline{F} \underset{\text{alg.}}{\subseteq} \overline{K}}_{\text{alg.}}$$

So every elt. of $\overline{K}$ is a root of some poly $/F$.

$\square$

Thm: Every field F has an alg. closure $\overline{F}$, which is unique up to isom.

Pf: see D&F Props. 30 & 31

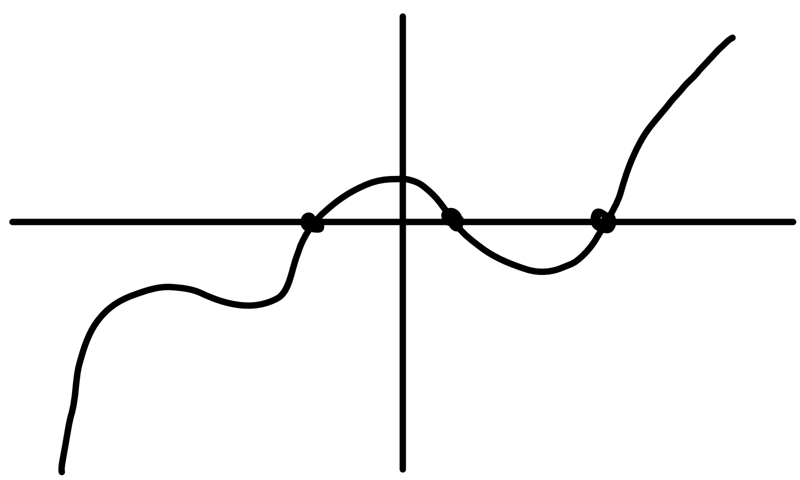
Fundamental Thm. of Algebra (Gauss): $\mathbb{C}$ is alg. closed

Cor: If $F \subseteq \mathbb{C}$, then $\overline{F} \subseteq \mathbb{C}$, so e.g. $\overline{\mathbb{Q}}$ = set of alg. numbers

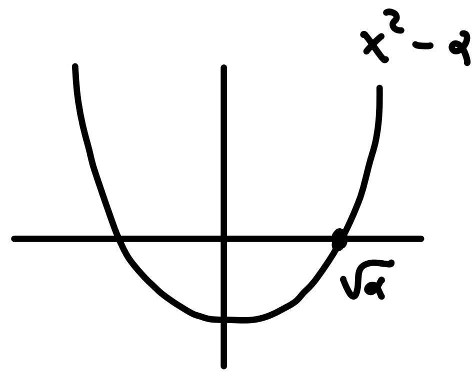
Pf sketch using Galois theory:

Two analytic consequences of the Intermediate Value Theorem

(A) Every odd degree poly. in $\mathbb{R}[x]$ has a root in $\mathbb{R}$



(B) Every $\alpha \in \mathbb{R}_{\geq 0}$ has a sqrt. $\sqrt{\alpha} \in \mathbb{R}_{\geq 0}$



Let $f(x) \in \mathbb{R}[x]$, f irred., $n := \deg f$.

WTS: f has a root in $\mathbb{C}$.

Let $K := Sp_{\mathbb{R}} f$

$$\begin{tikzcd}[column sep=small]
 & K(i) \arrow[dl, dash] \arrow[dr, dash, "1 \text{ or } 2"] & \\
\mathbb{C} \arrow[dr, dash, "2"'] & & K \arrow[dl, dash, "n"] \\
 & \mathbb{R} &
\end{tikzcd}$$

Galois theory gives us detailed information about intermediate fields.

In this case,

$$\begin{tikzcd}
K(i) \arrow[d, dash, "\text{power of } 2"] \\
\exists\, E \arrow[d, dash, "\text{odd}"] \\
\mathbb{R}
\end{tikzcd}$$

impossible by (A) unless $E = \mathbb{R}$

So we have

$$\begin{tikzcd}
 & & K(i) \arrow[d, dash, "2"] \\
 & \exists \arrow[r] \arrow[dd, bend right] & E_k \arrow[d, dash, dotted, "2"] \\
K(i) \arrow[d, dash, "\text{power of } 2"'] \arrow[r, squiggly, "\text{more Galois theory}"] & {} & \vdots \arrow[d, dash, "2"] \\
\mathbb{R} & & E_1 \arrow[d, dash, "2"] \\
 & & \mathbb{C} \arrow[d, dash, "2"] \\
 & & \mathbb{R}
\end{tikzcd}$$

impossible by (B) unless $\mathbb{C} = K(i)$

$\square$