Announcements

Midterm 1: Tonight 7:00-8:30pm Sidney Lu 1043
Reference sheet allowed

See last week's email for full policies

Friday's office hour moved to today @ 4:00-5:00pm (CAB 69B)

Midterm 1 review

Partial list of some things we know about:

Classes of integral domain

fields $\subseteq$ EDs $\subseteq$ PIDs $\subseteq$ UFDs $\subseteq$ int. doms.
(plus def'ns and examples)

Norms (Euclidean, or coming from $\mathbb{C}$)

Factorizations, gcds, primes, irreducibles, prime / max'l ideals

how to compute

relation btwn.

relation btwn.

Factorization in $\mathbb{Z}[i] \Leftrightarrow$ writing primes $\in \mathbb{N}$ as a^2+b^2
(Fermat's Theorem)

Polynomial rings

- Euclidean norm (if over field)
- R UFD $\Leftrightarrow$ R[x] UFD
- Irreducibitiy criteria
  - Gauss' Lemma
  - Test for roots
  - Reduction mod ideal
  - Rational root thm.
  - Eisenstein's criterion
  - Ad-hoc techniques (like plugging in $x+1$)

Linear algebra (enough to get by)

- Vector space (over a field), linear independence, span, basis, dimension (see §11.1)

Field theory

- Characteristic & prime subfield
- Field extension, simple ext'n, degree
- Construction of $F(\alpha)$ $\left(\cong F(x)/(m_{\alpha,F})\right)$
- Algebraic vs. transcendental
- Finite vs. infinite

Minimal poly and properties

Tower Law and consequences

Computations in $F(\alpha)$

Other suggestions

Look at lecture notes, hw problems, practice problems

Look at result statements in D&F

Understand all the "little pieces" and be able to fit them together

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Practice problems (pf. sketches posted on website)

9.3.4) Let $R = \mathbb{Z} + x\,\mathbb{Q}[x] \subseteq \mathbb{Q}[x]$

$$R = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_0 \in \mathbb{Z},\ a_i \in \mathbb{Q}\}$$

a) Prove that R: int. domain w/ units ± 1

Pf: R is a subring of $\mathbb{Q}[x]$ (closed under $+, -, \cdot$) so it has no zero-divisors, so is an int. dom.

Let $N: R \to \mathbb{Z}_{\geq 0}$
$f \mapsto \deg f$

$$N(fg) = N(f) + N(g)$$

All units must have norm 0, so must be a unit in $\mathbb{Z}$, so are ± 1

$\square$

b) Show that the irreds. in R are

$\{p : \text{prime in } \mathbb{Z}\} \cup \{f(x) \text{ irred. in } \mathbb{Q}[x], \text{ constant term } \pm 1\}$

Prove that these irreds. are prime in R

prime in $\mathbb{Z}$

Pf: If $p = fg$, $0 = N(p) = N(f) + N(g)$, so $f, g \in \mathbb{Z}$.
Since p is prime in $\mathbb{Z}$, either f or g is a unit.
If $f(x) \in \mathbb{Q}[x]$ is irred in $\mathbb{Q}[x]$ and has constant term ± 1, if $f = gh$, $g, h \in R$, g and h must have constant terms ± 1, so if they are nonunits they have norm ≥ 1. But then f is reducible in $\mathbb{Q}[x]$.

Conversely, if $f(x) \in R$ ($\deg f \geq 1$) is irred., then its constant term c is ± 1 (otherwise $f = p \frac{f(x)}{p}$, for any prime $p \in \mathbb{Z}$ dividing c, is a nontriv. factorization). If f is red in $\mathbb{Q}[x]$ i.e. $f(x) = g(x) h(x)$, where $g(x)$ has constant term g_0 and $h(x)$ has constant term $\pm g_0^{-1}$, then $f(x) = \tilde{g}(x) \tilde{h}(x)$ where $\tilde{g} := \frac{g}{g_0}$, $\tilde{h} : g_0 h \in R$.

Finally, if $f(x)$ is irred. in $\mathbb{Q}[x]$, it is prime in $\mathbb{Q}[x]$ since $\mathbb{Q}[x]$ is a PID. Therefore, f is prime in the subring R.
If $p \in \mathbb{Z}$ is prime in $\mathbb{Z}$, it is prime in R since

$R/(p) \cong \mathbb{Z}/p\mathbb{Z}$, which is an int. dom. $\square$

c) Show that x cannot be written as a product of irreds. in R (so R is not a UFD).

Pf: If $x = f_1 f_2 \cdots f_n$ is a product of irreds, then $1 = N(x) = N(f_1) + \cdots + N(f_n)$, so WLOG,

$N(f_1) = 1$, $N(f_2) = \cdots = N(f_n) = 0$. We have $f_1 = ax + b$,

but $b = 0$ since otherwise $f_1 \cdots f_n$ would have nonzero constant term. However, $ax = 2 \cdot \frac{a}{2} x$ is a nontriv. factorization, so f_1 is not irred. $\square$

d) Show that x is not prime in R, and describe the quotient ring $R/(x)$.

Pf: In an integral domain, prime $\Rightarrow$ irred.

We claim that

$$R/(x) = \{ \overline{a + bx} \mid a \in \mathbb{Z},\ b \in \mathbb{Q},\ 0 \le b < 1 \} \cong \mathbb{Z} + (\mathbb{Q}/\mathbb{Z})x$$

No two of these elements differ by a mult of x.

On the other hand, if $f(x) \in R$,

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n, \qquad a_0 \in \mathbb{Z},\ a_i \in \mathbb{Q}$$

$$= a_0 + a_1 x + x\underbrace{(a_2 x + \cdots + a_n x^{n-1})}_{\in R}$$

$$= a_0 + (a_1 - \lfloor a_1 \rfloor)x + x\underbrace{(\lfloor a_1 \rfloor + a_2 x + \cdots + a_n x^{n-1})}_{\in R}$$

"floor"

$$\mapsto \overline{a_0 + \underbrace{(a_1 - \lfloor a_1 \rfloor)}_{\in [0,1)} x}$$

$\square$

13.2.12: Suppose $[K:F]$ is a prime p. If, $F \subseteq E \subseteq K$, then $E = F$ or $E = K$.

Pf: By the Tower Law,

$$p = [K:F] = [K:E][E:F],$$

so either $[K:E] = p$, $[E:F] = 1$, in which case $E = F$, or $[K:E] = 1$, $[E:F] = p$, in which case $E = K$.

$\square$

Side note: In general, if $[K:F] = n$, then the values $[K:E]$ and $[K:F]$ must be factors of n. But unless one of them is 1, we can't say what E is.

We could also ask: If $[K:F]=mn$, does there always exist a field E, $F \subseteq E \subseteq K$ s.t. $[E:F]=m$?

Ans: no, but we need Galois theory!