Announcement

Please select pages for each problem for Gradescope submissions

Recall: last time we said that every elt. of $F(\alpha, \beta)$ is an F-linear comb. of $a_i b_j$, where $\{a_i\}$ (resp. $\{b_j\}$) form a basis for $F(\alpha,\beta)/F(\alpha)$ (resp $F(\alpha)/F$)

Tower Law: Let $F \subseteq K \subseteq L$. Then,

$$[L:F] = [L:K][K:F]$$

Example: $\overset{F}{\mathbb{Q}} \subseteq \overset{K}{\mathbb{Q}(\underset{\overset{\shortparallel}{\alpha^3}}{\sqrt{2}})} \subseteq \overset{L}{\mathbb{Q}(\underbrace{\sqrt[6]{2}}_{\alpha})}$

$\beta \in \mathbb{Q}(\sqrt[6]{2}) \quad \beta = a + b\alpha + c\alpha^2 + d\alpha^3 + e\alpha^4 + f\alpha^5$

$$= (a + d\sqrt{2}) + (b + e\sqrt{2})\alpha + (c + f\sqrt{2})\alpha^2$$

Basis for K/F : $1, \sqrt{2}$

Basis for L/K : $1, \alpha, \alpha^2$

Basis for L/F : $1, \alpha, \alpha^2, \underset{\overset{\shortparallel}{\sqrt{2}}}{\alpha^3}, \underset{\overset{\shortparallel}{\alpha\sqrt{2}}}{\alpha^4}, \underset{\overset{\shortparallel}{\alpha^2\sqrt{2}}}{\alpha^5}$

Pf: First assume RHS is finite.

$$n := [K:F] \quad \text{basis: } \alpha_1, \ldots, \alpha_n \in K$$

$$m := [L:K] \quad \text{basis: } \beta_1, \ldots, \beta_m \in L$$

We claim that $\{\gamma_{ij} := \alpha_i \beta_j \in L\}$ forms an F-basis for L.

Let $l \in L$. Since $\{\beta_1, \ldots, \beta_m\}$ basis for L/K,

$$l = k_1\beta_1 + \cdots + k_m\beta_m, \quad k_i \in K \quad \text{(unique!)}$$

Since $\{\alpha_1, \ldots, \alpha_n\}$ basis for K/F,

$$k_i = f_{i1}\alpha_1 + \cdots + f_{in}\alpha_n, \quad f_{ij} \in F \quad \text{(unique!)}$$

So

$$l = f_{11}\beta_1\alpha_1 + f_{12}\beta_1\alpha_2 + \cdots + f_{nm}\beta_n\alpha_m \quad \text{(unique!)}$$

Now, if RHS is infinite, LHS is also infinite since

$$[L:F] \geq [L:K] \quad \text{and} \quad [L:F] \geq [K:F]$$

$\square$

Cor: $F \subseteq K \subseteq L$.

a) If L/K and K/F are both finite, so is L/F

b) If L/K and K/F are both algebraic, so is L/F

Pf: a) follows from the Tower Law.

b) Let $\beta \in L$, and consider

$$m_{\beta,K}(x) = x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \in K[x].$$

Since simple alg. ext'ns are finite (w/ degree equal to deg. min'l poly.), $F(\beta)/F$ is finite since

$$F \subseteq F(\alpha_0) \subseteq F(\alpha_0, \alpha_1) \subseteq \cdots \subseteq F(\alpha_0, \ldots, \alpha_n) \subseteq F(\alpha_0, \ldots, \alpha_n, \beta)$$

are simple, alg. ext'ns. Thus β is alg. $/F$ $\forall \beta \in L$, so L is alg $/F$. $\square$

Surprising consequences such as:

Ex: $\sqrt{2} \notin \mathbb{Q}(\sqrt[3]{2})$

Pf: $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$ since $x^n - 2$ is irred.

If $\sqrt{2} \in \mathbb{Q}(\sqrt[3]{2})$, then $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[3]{2})$ and

$$3 = [\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}(\sqrt{2})]\underbrace{[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}]}_{2},$$ a contradiction $\square$

Def: If $K_1, K_2 \subseteq L$, the <u>composite</u> K_1K_2 of K_1 and K_2 is the smallest field containing K_1 and K_2.

E.g. a) $F(\alpha)F(\beta) = F(\alpha, \beta)$

b) $\underbrace{\mathbb{Q}(\sqrt{2})\mathbb{Q}(\sqrt[3]{2})}_{K} = \mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) \underset{*}{=} \mathbb{Q}(\sqrt[6]{2})$ in $\mathbb{C}$

Pf 1 of $*$: $\sqrt{2}, \sqrt[3]{2} \in \mathbb{Q}(\sqrt[6]{2})$

$$\sqrt[6]{2} = \sqrt{2}/\sqrt[3]{2} \in \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$$

Pf 2 of $*$: $\sqrt{2}, \sqrt[3]{2} \in \mathbb{Q}(\sqrt[6]{2})$

$$[\mathbb{Q}(\sqrt[6]{2}):\mathbb{Q}] = 6 \mid [\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}):\mathbb{Q}],$$

since 2 and 3 divide it

so $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}(\sqrt{2}, \sqrt[3]{2})] = 1 \implies$ they are equal

Prop: Let K_1/F, K_2/F be finite ext'ns w/ $K_1, K_2 \in L$.

a) $[K_1K_2 : K_2] \leq [K_1 : F]$

b) $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$

Pf: Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis for K_1 over F.

Let $K = \{f_1\alpha_1 + \cdots + f_n\alpha_n \mid f_i \in K_2\}$

We have $K_1 \subseteq K$, $K_2 \subseteq K$, and $\dim_{K_2} K \leq n$, so if it's a field it is K_1K_2, and a) will hold.

Closed under $+, -$: yes, since K is a v.s.

Closed under $\cdot$:

Since $\alpha_1, \dots, \alpha_k$ is an F-basis for K_1, write

$$\alpha_i\alpha_j = \sum_k \underbrace{h_k}_{\in F \subseteq K_2} \alpha_k$$

Then,

$$(f_1\alpha_1 + \cdots + \overset{\in K_2}{f_n}\alpha_n)(g_1\alpha_1 + \cdots + \overset{\in K_2}{g_n}\alpha_n)$$

$$= \sum_{i,j,k} \underbrace{f_i g_j}_{\in K_2} \underbrace{\alpha_i\alpha_j}_{\in K_1} = \sum_{i,j,k} f_i g_j h_k \alpha_k = \sum_k \underbrace{\left(\sum_{i,j} f_i g_j h_k\right)}_{\in K_2} \alpha_k$$

Inverses: Let $\gamma \in K \setminus \{0\}$, and consider the K_2-linear transformation

$$T_\gamma : K \to K \qquad \left(\begin{array}{l}\text{additive gp. homom.,}\\ \text{but } \underline{\text{not}} \text{ ring homom.}\end{array}\right)$$
$$a \mapsto a\gamma$$

Since L is an integral domain,

$\ker(T_\gamma) = \{0\}$, so by the rank-nullity theorem,

$\dim \operatorname{im} T_\gamma + \underbrace{\dim \ker T_\gamma}_{0} = n$, so T_γ is onto.

Thus γ has inverse $T_\gamma^{-1}(1) \in K$.

b) Using the Tower Law,

$$[K_1 : F][K_2 : F] \geq [K_1K_2 : K_2][K_2 : F] = [K_1K_2 : F]$$

$\square$

Alternate pf (see D&F): Finite ext'ns are interated simple extensions. Prove a) for simple ext'ns by considering degrees of min'l polys, and use induction for the general case