Math 418, Spring 2025 – Homework 9

Due: Friday, April 18th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition*. All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. **Dummit and Foote #14.6.2a** *Determine the Galois group of the polynomial* $f(x) = x^3 - x^2 - 4$

2. **Dummit and Foote #14.6.10** *Determine the Galois group of* $x^5 + x - 1$. *(Hint: see D & F Proposition 14.21*

3. *Let* $p_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k$ *be the* power sum symmetric function, *and let* $e_k(x_1, \ldots, x_n) = \sum_{i_1 < \ldots < i_k} x_{i_1} \cdots x_{i_k}$ *be the elementary symmetric function. Let*

$$E(t) = \sum_{r=0}^{\infty} e_r(x_1, \ldots, x_n)t^r, \qquad P(t) = \sum_{r=1}^{\infty} p_r(x_1, \ldots, x_n)t^{r-1}.$$

Prove that

$$E(t) = \prod_{i=1}^{n}(1 + x_i t), \qquad P(t) = \sum_{i=1}^{n} \frac{x_i}{1 - x_i t} = \sum_{i=1}^{n} \frac{d}{dt} \ln \frac{1}{1 - x_i t}.$$

4. **Dummit and Foote #14.6.22** *Let* $f(x)$ *be a monic polynomial of degree* n *with roots* $\alpha_1, \ldots, \alpha_n$. *Let* e_i *be the elementary symmetric function of degree* i *in the roots and define* $e_i = 0$ *for* $i > n$. *Let* $p_i = \alpha_1^i + \cdots + \alpha_n^i, i \geq 0$, *be the sum of the ith powers of the roots of* $f(x)$ *Prove Newton's formulas:*

$$p_n - e_1 p_{n-1} + e_2 p_{n-2} + \cdots + (-1)^{n-1} e_{n-1} p_1 + (-1)^n n e_n = 0.$$

(Hint: use solution to previous problem)

5. **Dummit and Foote #14.7.1** *Use Cardano's Formulas to solve the equation* $f(x) = x^3 + x^2 - 2 = 0$. *In particular show that the equation has the real root*

$$\frac{1}{3}\left(\sqrt[3]{26 + 15\sqrt{3}} + \sqrt[3]{26 - 15\sqrt{3}} - 1\right).$$

Show directly that the roots of this cubic are $1, -1 \pm i$. *Explain this by proving that*

$$\sqrt[3]{26 + 15\sqrt{3}} = 2 + \sqrt{3}, \qquad \sqrt[3]{26 - 15\sqrt{3}} = 2 - \sqrt{3}$$

so that

$$\sqrt[3]{26 + 15\sqrt{3}} + \sqrt[3]{26 - 15\sqrt{3}} = 4.$$

6. **Dummit and Foote #14.7.17** *Let* $D \in \mathbb{Z}$ *be a squarefree integer and let* $a \in \mathbb{Q}$ *be a nonzero rational number. Show that* $\mathbb{Q}(\sqrt{a\sqrt{D}})$ *cannot be a cyclic extension of degree 4 over* $\mathbb{Q}$ *(i.e.* $Gal(\mathbb{Q}(\sqrt{a\sqrt{D}})/\mathbb{Q})$ *cannot be* $\mathbb{Z}/4\mathbb{Z}$*).*