Math 418, Spring 2025 – Homework 8

Due: Wednesday, April 9th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. **Dummit and Foote #14.2.6:** *Let* $K = \mathbb{Q}(\sqrt[8]{2}, i)$ *and let* $F_1 = \mathbb{Q}(i), F_2 = \mathbb{Q}(\sqrt{2}), F_3 = \mathbb{Q}(\sqrt{-2})$. *Prove that* $Gal(K/F_1) = \mathbb{Z}/8\mathbb{Z}, Gal(K/F_2) = D_8, Gal(K/F_3) = Q_8$. *(Hint: use the example in Section 14.2, and the diagrams on pages 580-1)*

2. **Dummit and Foote #14.2.7:** *Determine all the subfields of the splitting field of* $x^8 - 2$ *which are Galois over* $\mathbb{Q}$. *(Hint: use the example in Section 14.2, and the diagrams on pages 580-1)*

3. **Dummit and Foote #14.2.14:** *Show that* $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ *is a cyclic quartic field, i.e., is a Galois extension of degree 4 with cyclic Galois group.*

4. *Let* K/F *be a Galois extension of degree* n *with* $G = Gal(K/F)$. *For* $\alpha \in K$, *define the norm and trace of* α *by*

$$N_{K/F}(\alpha) := \prod_{\sigma \in G} \sigma(\alpha), \qquad \text{and} \qquad Tr_{K/F}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha).$$

   Let $m_{\alpha,F}(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$.

   (a) Show that $N_{K/F}(\alpha) = (-1)^n a_0^{n/d}$ and $Tr_{K/F}(\alpha) = -\frac{n}{d} a_{d-1}$.

   (b) Show that

$$N_{K/F}(\alpha\beta) = N_{K/F}(\alpha)N_{K/F}(\beta) \qquad \text{and} \qquad Tr_{K/F}(\alpha+\beta) = Tr_{K/F}(\alpha) + Tr_{K/F}(\beta).$$

   (c) Show that $N_{K/F}(a\alpha) = a^n N_{K/F}(\alpha)$ and $Tr_{K/F}(a\alpha) = aTr_{K/F}(\alpha)$ for all $a \in F$, In particular show that $N_{K/F}(a) = a^n$ and $Tr_{K/F}(a) = na$ for all $a \in F$.

5. **Dummit and Foote #14.5.3:** *Determine the quadratic equation satisfied by the period* $\alpha = \zeta_5 + \zeta_5^{-1}$ *of the 5th root of unity* ζ_5. *Determine the quadratic equation satisfied by* ζ_5 *over* $\mathbb{Q}(\alpha)$ *and use this to explicitly solve for the 5th root of unity.*

6. **Dummit and Foote #14.5.7:** *Show that complex conjugation restricts to the automorphism* $\sigma_{-1} \in Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ *of the cyclotomic field of nth roots of unity. Show that the field* $K^+ = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ *is the subfield of real elements in* $K = \mathbb{Q}(\zeta_n)$*, called the maximal real subfield of* K*.*

7. **Dummit and Foote #14.5.10:** *Prove that* $\mathbb{Q}(\sqrt[3]{2})$ *is not a subfield of any cyclotomic field over* $\mathbb{Q}$*.*