Math 418, Spring 2025 – Homework 7

Due: Wednesday, April 2nd, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. **Dummit and Foote #14.3.3:** *Prove that an algebraically closed field must be infinite.*

2. **Dummit and Foote #14.3.4:** *Construct the finite field of 16 elements and find a generator for the multiplicative group. How many generators are there?*

3. **Dummit and Foote #14.3.8:** *Determine the splitting field of the polynomial* $f(x) = x^p - x - a$ *over* $\mathbb{F}_p$ *where* $a \neq 0, a \in F_p$. *Show explicitly that the Galois group is cyclic.*

4. **Dummit and Foote #14.4.2:** *Find a primitive element for* $K := \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ *over* $\mathbb{Q}$.

5. **Dummit and Foote #14.4.3:** *Let* F *be a field contained in the ring* $Mat_n(\mathbb{Q})$ *of* $n \times n$ *matrices over* $\mathbb{Q}$. *Here,* $\mathbb{Q} \subseteq Mat_n(\mathbb{Q})$ *is identified with the scalar diagonal matrices by the inclusion*

$$q \mapsto qI = \begin{bmatrix} q & & & \\ & q & & \\ & & \cdots & \\ & & & q \end{bmatrix}.$$

Prove that $[F : \mathbb{Q}] \leq n$. *(I do have a hint for this one, if you ask)*