Math 418, Spring 2025 – Homework 6

Due: Friday, March 14th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. **Dummit and Foote #13.5.3:** *Prove that d divides n if and only if $x^d - 1$ divides $x^n - 1$.* (Hint: if $n = qd + r$, then $x^n - 1 = (x^{qd+r} - x^r) + (x^r - 1)$)

2. **Dummit and Foote #13.5.6:** *Prove that $x^{p^n-1} - 1 = \prod_{\alpha \in \mathbb{F}_{p^n}^\times}(x - \alpha)$. Conclude that $\prod_{\alpha \in \mathbb{F}_{p^n}^\times} \alpha = (-1)^{p^n}$ so the product of the nonzero elements of a finite field is $+1$ if $p = 2$ and -1 if p is odd. For p odd and $n = 1$ derive Wilson 's Theorem: $(p-1)! = -1(\bmod p)$.*

3. **Dummit and Foote #13.6.2:** *Let ζ_n be a primitive nth root of unity and let d be a divisor of n. Prove that ζ_n^d is a primitive (n/d)th root of unity.*

4. **Dummit and Foote #13.6.3:** *Prove that if a field contains the nth roots of unity for n odd then it also contains the $2n$th roots of unity.*

5. **Dummit and Foote #13.6.7:** *Use the Mobius Inversion formula indicated in Section 14.3 to prove*
$$\Phi_n(x) = \prod_{d|n}(x^d - 1)^{\mu(n/d)}.$$

6. **Dummit and Foote #14.1.3:** *Determine the fixed field of complex conjugation on $\mathbb{C}$.*

7. **Dummit and Foote #14.1.5:** *Determine the automorphisms of the extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ explicitly.* (Hint: Use Dummit & Foote Proposition 14.5)