Math 418, Spring 2025 – Homework 4

Due: Wednesday, February 26th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. **Dummit and Foote #13.2.1:** *Let $\mathbb{F}$ be a finite field of characteristic p. Prove that $|\mathbb{F}| = p^n$ for some positive integer n.*

   Solution. Since $\mathbb{F}$ has characteristic p, the prime subfield of $\mathbb{F}$ is isomorphic to $\mathbb{F}_p$. Therefore, $F/\mathbb{F}_p$ is a field extension, so $\mathbb{F}$ is a vector space over $\mathbb{F}_p$, and so $\mathbb{F} = \{a_1 v_1 + \ldots + a_n v_n | a_n \in \mathbb{F}_p\}$ has order p^n.

2. **Dummit and Foote #13.2.4:** *Determine the degree over $\mathbb{Q}$ of $2+\sqrt{3}$ and of $1+\sqrt[3]{2}+\sqrt[3]{4}$.*

   Solution. For the first problem, since $2+\sqrt{3} \in \mathbb{Q}(\sqrt{3})$ and $\sqrt{3} \in \mathbb{Q}(2+\sqrt{3})$, we have $\mathbb{Q}(2+\sqrt{3}) = \mathbb{Q}(\sqrt{3})$. By Proposition 11, $\sqrt{3}$, the extension $\mathbb{Q}(\sqrt{3})/\mathbb{Q}$, and $2+\sqrt{3}$ all have the same degree, and since $x^2 - 3$ is the minimal polynomial for $\sqrt{3}$, this degree is 2.

   We approach the second problem similarly. Let $\theta = 1+\sqrt[3]{2}+\sqrt[3]{4}$. $\theta \in \mathbb{Q}(\sqrt[3]{2})$ since $\sqrt[3]{4} = (\sqrt[3]{2})^2$. On the other hand, $\theta^2 = 5 + 4\sqrt[3]{2} + 3\sqrt[3]{4}$, so $\sqrt[3]{2} = \theta^2 - 3\theta - 2 \in \mathbb{Q}(\theta)$ Therefore, θ has the same degree as $\sqrt[3]{2}$ i.e. 3.

   Alternatively, we can show the containments in one direction, and use the Tower Law to show the extension degrees must be the same in each case.

3. **Dummit and Foote #13.2.5:** *Let $F = \mathbb{Q}(i)$. Prove that $x^3 - 2$ and $x^3 - 3$ are irreducible over F.*

   Solution. We'll consider the polynomial $p(x) = x^3 - 2$, and the other one is similar. By Proposition 11, we can prove the result by showing that $[\mathbb{Q}(i, \sqrt[3]{2}) : \mathbb{Q}(i)] = 3$ (see also Lemma 16). $p(x)$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion, so it's the minimal polynomial for $\sqrt[3]{2}$, and by Proposition 11, $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$. Also, $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ since i has minimal polynomial $x^2 + 1$. The Tower Law then says

   $$[\mathbb{Q}(i, \sqrt[3]{2}) : \mathbb{Q}(i)][\mathbb{Q}(i) : \mathbb{Q}] = [\mathbb{Q}(i, \sqrt[3]{2}) : \mathbb{Q}] = [\mathbb{Q}(i, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})][\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}],$$

so

$$[\mathbb{Q}(i, \sqrt[3]{2}) : \mathbb{Q}(i)] = \frac{3[\mathbb{Q}(i, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})]}{2}$$

is a multiple of 3.

4. **Dummit and Foote #13.2.7:** *Prove that* $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. *Conclude that* $[\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}] = 4$. *Find an irreducible polynomial satisfied by* $\sqrt{2}+\sqrt{3}$.

   Solution. Since $\theta := \sqrt{2}+\sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$, we have containment one way. For the other direction, note that $\theta^3 = 11\sqrt{2} + 9\sqrt{3}$, so both $\sqrt{2} = \frac{1}{2}(\theta^3 - 9\theta)$ and $\sqrt{3} = -\frac{1}{2}(\theta^3 - 11\theta)$ are in $\mathbb{Q}(\theta)$.

   By Corollary 15, $[\mathbb{Q}(\theta) : \mathbb{Q}(\sqrt{2})] \leq 2$, and since $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$ this degree must equal 2. Therefore, by the tower law,

   $$[\mathbb{Q}(\theta) : \mathbb{Q}] = [\mathbb{Q}(\theta) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4.$$

   Finally, we compute $\theta^2 = 5+2\sqrt{6}$ and $\theta^4 = 49+20\sqrt{6}$, and conclude that $\theta^4 - 10\theta^2 + 1 = 0$.

5. **Dummit and Foote #13.3.2:** *Prove that Archimedes' construction actually trisects the angle* θ. *(See the book for the construction).*

   Solution. Let ϕ be the third angle of the triangle lying within the circle, ϵ be the angle supplementary to β, and η be the remaining angle of the other triangle. We have $\beta = \gamma$ and $\alpha = \eta$ since these pairs of angles are each part of the same isosceles triangle. Adding up the angles in the two triangles gives $\epsilon + 2\alpha = 180°$ and $\phi + 2\beta = 180°$. Decomposing straight line angles gives $\epsilon + \beta = 180°$ and $\alpha + \phi + \theta = 180°$; in particular, $\beta = 2\alpha$. Solving this last equation for θ and substituting, we get

   $$\theta = 180° - \phi - \alpha = 2\beta - \alpha = 3\alpha.$$

6. **Dummit and Foote #13.3.4:** *The construction of the regular 7-gon amounts to the constructibility of* $\cos(2\pi/7)$. *We shall see later (Section 14.5 and Exercise 2 of Section 14.7) that* $\alpha = 2\cos(2\pi/7)$ *satisfies the equation* $p(x) = x^3 + x^2 - 2x - 1 = 0$. *Use this to prove that the regular 7-gon is not constructible by straightedge and compass.*

   Solution. This problem amounts to showing that the degree of $\cos(2\pi/7)$ over $\mathbb{Q}$ is not a power of 2, for which it suffices to show that $p(x)$ is irreducible. Since $p(x)$ is cubic, by Propositions 9 and 10 of Chapter 9, $p(x)$ is reducible if and only if it has a root. By the rational root theorem, such a root must be ± 1, and plugging in shows neither is a root.

7. **Dummit and Foote #13.3.5:** *Use the fact that* $\alpha = 2\cos(2\pi/5)$ *satisfies the equation* $x^2 + x - 1 = 0$ *to conclude that the regular* $5-$*gon is constructible by straightedge and compass.*

Solution. Using the quadratic formula, $\alpha = \frac{-1 \pm \sqrt{5}}{2}$, which is constructible since the constructible numbers form a field which is closed under taking square roots. Constructing 1 and $\cos\theta$ allows us to construct the angle θ. Finally, the interior angle of a pentagon is $3\pi/5$, which is complimentary to $2\pi/5$.