Math 418, Spring 2025 – Homework 10

Due: Wednesday, May 7th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. Let k be an algebraically closed field, and consider the polynomial ring $k[x, y]$.
   (a) Let V be the x-axis, i.e. $V = V(y)$. Prove that V is irreducible. [Hint: Show a prime ideal is radical.]
   (b) Prove that $V = V(x - y)$ is irreducible.
   (c) Prove that $S = \{(a, a) \in k^2 | a \neq 1\}$ is *not* an algebraic variety if $k = \mathbb{C}$.
   (d) What is the decomposition of $V = V(x^2 - y^2)$ into irreducibles? **Warning:** The answer depends on k!
2. **Dummit and Foote #15.1.2** Show that each of the following rings are not Noetherian by exhibiting an explicit infinite increasing chain of ideals:
   (a) the ring of continuous real valued functions on $[0, 1]$
   (b) the ring of all functions from any infinite set X to $\mathbb{Z}/2\mathbb{Z}$.
3. **Dummit and Foote #15.1.20** *If f and g are irreducible polynomials in $k[x, y]$ that are not associates (do not divide each other), show that $V((f, g))$ is either $\emptyset$ or a finite set in k^2.* [Hint: If $(f, g) \neq (1)$, show (f, g) contains a nonzero polynomial in $k[x]$ (and similarly a nonzero polynomial in $k[y]$) by letting $R = k[x], F = k(x)$, and applying Gauss's Lemma to show f and g are relatively prime in $F[y]$.]
4. **Dummit and Foote #15.2.2** Let I and J be ideals in the ring R. Prove the following statements:
   (a) If $I^k \subseteq J$ for some $k \geq 1$, then $\sqrt{I} \subseteq \sqrt{J}$.
   (b) If $I^k \subseteq J \subseteq I$ for some $k \geq 1$, then $\sqrt{I} = \sqrt{J}$.
   (c) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
   (d) $\sqrt{\sqrt{I}} = \sqrt{I}$.

(e) $\sqrt{I}+\sqrt{J} \subseteq \sqrt{I+J}$ and $\sqrt{I+J}=\sqrt{\sqrt{I}+\sqrt{J}}$.

5. **Dummit and Foote #15.2.3** *Prove that the intersection of two radical ideals is again a radical ideal.*

6. **Dummit and Foote #15.2.5** *If $I=(xy,(x-y)z) \subseteq k[x,y,z]$ prove that $\sqrt{I}=(xy,xz,yz)$. For this ideal prove* directly *that $V(I)=V(\sqrt{I})$, that $V(I)$ is not irreducible, and that $\sqrt{I}$ is not prime.*

7. **Cox, Little, and O'Shea #8.2.3** *In this exercise, we will study how lines in $\mathbb{R}^n$ relate to points at infinity in $P^n(\mathbb{R})$. We will use the decomposition $P^n(\mathbb{R})=\mathbb{R}^n \cup P^{n-1}(\mathbb{R})$. Given a line L in $\mathbb{R}^n$, we can parametrize L by the formula $a+bt$, where $a \in L$ and b is a nonzero vector parallel to L. In coordinates, we write this parametrization as $(a_1+b_1t,\ldots,a_n+b_nt)$.*

   (a) We can regard L as lying in $P^n(\mathbb{R})$ using the homogeneous coordinates
   $$[1 : a_1+b_1t : \cdots a_n+b_nt].$$
   To find out what happens as $t \to \pm\infty$ divide by t to obtain
   $$\left[\frac{1}{t} : \frac{a_1}{t}+b_1 : \cdots \frac{a_n}{t}+b_n\right].$$
   As $t \to \pm\infty$, what point of $H=P^{n-1}(\mathbb{R})$ do you get?

   (b) The line L will have many parametrizations. Show that the point of $P^{n-1}(\mathbb{R})$ given by part (a) is the same for all parametrizations of L. Hint (from the book): Two nonzero vectors are parallel if and only if one is a scalar multiple of the other.

   (c) Parts (a) and (b) show that a line L in $\mathbb{R}^n$ has a well-defined point at infinity in $H=P^{n-1}(\mathbb{R})$. Show that two lines in $\mathbb{R}^n$ are parallel if and only if they have the same point at infinity.

8. **Cox, Little, and O'Shea #8.2.7** *In this exercise, we will study when a nonhomogeneous polynomial has a well-defined zero set in $P^n(k)$. Let k be an infinite field. We will show that if $f \in k[x_0,...,x_n]$ is not homogeneous, but f vanishes on all homogeneous coordinates of some $p \in P^n(k)$, then each of the homogeneous components f_i of f (see Definition 6 of Chapter 7, Section 1) must vanish at p.*

   (a) Write f as a sum of its homogeneous components $f=\sum_i f_i$. If $p=(a_0,\ldots,a_n)$ then show that
   $$f(\lambda a_0,\ldots,\lambda a_n)=\sum_i f_i(\lambda a_0,\ldots,\lambda a_n)=\sum_i \lambda^i f(a_0,\ldots,a_n)$$

   (b) Deduce that if f vanishes for all nonzero $\lambda \in k$, then $f_i(a_0,\ldots,a_n)=0$ for all i