Math 418, Spring 2025 – Homework 10

Due: Wednesday, May 7th, at 9:00am via Gradescope.

Instructions: Students should complete and submit all problems. Textbook problems are from Dummit and Foote, *Abstract Algebra, 3rd Edition.* All assertions require proof, unless otherwise stated. Typesetting your homework using LaTeX is recommended, and will gain you 1 bonus point per assignment.

1. Let k be an algebraically closed field, and consider the polynomial ring $k[x, y]$.

   (a) Let V be the x-axis, i.e. $V = V(y)$. Prove that V is irreducible. [Hint: Show a prime ideal is radical.]

   Solution. If I is a prime ideal, then if $a \cdots a = a^n \in I$, then a or a or $\ldots$ or a is in I, so $a \in I$ and so I is radical.

   y is irreducible since it is degree 1, so it is prime since $k[x, y]$ is a UFD. Therefore, (y) is a prime ideal, so by the bijection proved in class, $V(y)$ is irreducible.

   (Direct proof: Suppose $V = V_1 \cup V_2$. Since $I := (y)$ is radical (by the above), $I(V) = I$, so $I = I(V_1) \cap I(V_2)$. Since I is prime, $I = I(V_1)$ or $I(V_2)$, say $I = V(I_1)$, and then $V_1 = V(I(V_1)) = V(I) = V$.)

   (b) Prove that $V = V(x - y)$ is irreducible.

   Solution. Similarly, $x - y$ is irreducible since it is degree 1, so it is prime since $k[x, y]$ is a UFD. Therefore, $(x - y)$ is a prime ideal, so by the bijection proved in class, $V(x - y)$ is irreducible.

   (c) Prove that $S = \{(a, a) \in k^2 | a \neq 1\}$ is *not* an algebraic variety if $k = \mathbb{C}$.

   Solution. Let $V = \{(a, a) \in k^2 | a \in k\}$. then this is a variety with $I(V) = (x - y)$. If S is a variety, we have $V(I(S)) = S$. Let $f(x, y) \in I(S)$, and let $g(x) = f(x, x)$. Since $f \in I(V)$, $f(a, a) = 0$ for all $a \neq 0$, so $g(x)$ has roots at all $a \neq 0$. This is infinitely many roots and g is a polynomial, so $g = 0$, and so $f(0, 0) = 0$, so $f \in I(V)$. This means that $I(S) = I(V)$, and $V(I(S)) \neq S$, meaning that S is not a variety.

   (d) What is the decomposition of $V = V(x^2 - y^2)$ into irreducibles? **Warning:** The answer depends on k!

   Solution. We have $x^2 - y^2 = (x + y)(x - y)$, so we have $V = V(x + y) \cup V(x - y)$. However, if char $k = 2$, then $x + y = x - y$, so in that case we simply have $V = V(x + y)$.

2. **Dummit and Foote #15.1.2** Show that each of the following rings are not Noetherian by exhibiting an explicit infinite increasing chain of ideals:

   (a) the ring of continuous real valued functions on $[0,1]$
   Solution. Let

   $$I_j = \left\{\text{functions } f : [0,1] \to \mathbb{R} | f(x) = 0 \text{ for all } x \leq \frac{1}{j}\right\}.$$

   A direct check shows that this is an ideal, and we have

   $$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots.$$

   Alternate solution: Let I_j be the principal ideal generated by $f_j(x) = \sqrt[2^j]{x}$. On the interval $[0,1]$, this is a continuous real-valued function, and $f_j = f_{j+1}^2 \in (f_{j+1})$, so we have $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$. On the other hand, if $f_{j+1} \in (f_j)$, then $f_{j+1} = g f_j$ for some g in our ring. However, $\frac{f_{j+1}}{f_j} = \frac{1}{\sqrt[2^j]{x}}$ is not defined at 0, and in fact its limit as x approaches 0 is ∞, so no such g exists.

   (b) the ring of all functions from any infinite set X to $\mathbb{Z}/2\mathbb{Z}$.
   Solution. Let $a_1, a_2, \ldots$ be distinct elements of X. Let

   $$I_j = \{\text{functions } f : X \to \mathbb{Z}/2\mathbb{Z} | f(a_i) = 0 \text{ for all } i \geq j\}.$$

   A direct check shows that this is an ideal, and we have

   $$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots.$$

3. **Dummit and Foote #15.1.20** *If f and g are irreducible polynomials in $k[x,y]$ that are not associates (do not divide each other), show that $V((f,g))$ is either $\emptyset$ or a finite set in k^2.* [Hint: If $(f,g) \neq (1)$, show (f,g) contains a nonzero polynomial in $k[x]$ (and similarly a nonzero polynomial in $k[y]$) by letting $R = k[x], F = k(x)$, and applying Gauss's Lemma to show f and g are relatively prime in $F[y]$.]

   Solution. Use the hint. If $(f,g) = (1)$, then $V((f,g)) = \emptyset$. Otherwise, let $R = k[x], F = k(x)$, and consider f and g as elements of both $R[y] = k[x,y]$ and $F[y]$. R is a UFD, so by Gauss' lemma, f and g are irreducible in $F[y]$ since they are irreducible in $R[y]$. Since f and g are irreducible nonassociates, they are relatively prime, and since $F[y]$ is a Euclidean domain we have $fa + gb = 1$ for some $a, b \in F[y]$. Multiplying by a large enough power of x, we obtain $f\tilde{a} + g\tilde{b} \in k[x]$ for some $\tilde{a}, \tilde{b} \in k[x,y]$; in other words (f,g) contains an element $p \in k[x]$. By a similar argument, (f,g) contains an element $q \in k[y]$. Every element of $V((f,g))$ must be a root of p and q, so must be of the form (a,b) with $p(a) = q(b) = 0$. Since p and q are one-variable polynomials, they have finitely many roots, so $V((f,g))$ is finite.

4. **Dummit and Foote #15.2.2** Let I and J be ideals in the ring R. Prove the following statements:

(a) If $I^k \subseteq J$ for some $k \geq 1$, then $\sqrt{I} \subseteq \sqrt{J}$.
Solution. If $x \in \sqrt{I}$, $x^n \in I$ for some n, so since $I^k \subseteq J$, $x^{kn} \in J$. Therefore, $x \in \sqrt{J}$.

(b) If $I^k \subseteq J \subseteq I$ for some $k \geq 1$, then $\sqrt{I} = \sqrt{J}$.
Solution. Applying the previous part twice, we have $\sqrt{I} \subseteq \sqrt{J} \subseteq \sqrt{I}$, so $\sqrt{I} = \sqrt{J}$.

(c) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
Solution. If $x \in \sqrt{IJ}$, $x^n \in IJ \subseteq I \cap J$ for some n, so $x \in \sqrt{I \cap J}$. If $y \in \sqrt{I \cap J}$, $y^n \in I \cap J$ for some n, so $y^n \in I$ and $y^n \in J$, and $x \in \sqrt{I} \cap \sqrt{J}$. If $z \in \sqrt{I} \cap \sqrt{J}$, then for some m, n, $z^m \in I, z^n \in J$. Therefore, $z^{m+n} \in IJ$, so $z \in \sqrt{IJ}$.

(d) $\sqrt{\sqrt{I}} = \sqrt{I}$.
Solution. If $x \in \sqrt{\sqrt{I}}$, for some m, $x^m \in \sqrt{I}$, so for some n, $x^{mn} = (x^m)^n \in I$, so $x \in \sqrt{I}$. Conversely, every ideal is contained in its radical.

(e) $\sqrt{I} + \sqrt{J} \subseteq \sqrt{I+J}$ and $\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}}$.
Solution. If $z \in \sqrt{I} + \sqrt{J}$, we have $z = x + y$ with $x \in \sqrt{I}, y \in \sqrt{J}$. For some m, n, $x^m \in I, y^n \in J$, so $(x+y)^{m+n} = x^{m+n} + x^{m+n-1}y + \cdots + x^m y^n + \cdots + y^{m+n} \in I + J$ since every term has a factor of x^m or y^n. Thus, $\sqrt{I} + \sqrt{J} \subseteq \sqrt{I+J}$. By part a, $\sqrt{\sqrt{I} + \sqrt{J}} \subseteq \sqrt{I+J}$. Conversely, $I \subseteq \sqrt{I}, J \subseteq \sqrt{J}$, so $I + J \subseteq \sqrt{I} + \sqrt{J}$ and $\sqrt{I+J} \subseteq \sqrt{\sqrt{I} + \sqrt{J}}$.

5. **Dummit and Foote #15.2.3** *Prove that the intersection of two radical ideals is again a radical ideal.*

Solution. Let I and J be radical ideals, and let $x^n \in I \cap J$. Then $x^n \in I$, so since I is radical, $x \in I$. Similarly, $x^n \in J$, so since J is radical $x \in J$. Therefore, $x \in I \cap J$, so $I \cap J$ is radical.

6. **Dummit and Foote #15.2.5** *If* $I = (xy, (x-y)z) \subseteq k[x, y, z]$ *prove that* $\sqrt{I} = (xy, xz, yz)$. *For this ideal prove* directly *that* $V(I) = V(\sqrt{I})$, *that* $V(I)$ *is not irreducible, and that* $\sqrt{I}$ *is not prime.*

Solution. $z^2 \cdot xy + xz(x-y)z = x^2z^2 \in I$, so $xz \in \sqrt{I}$. Since $xy, (x-y)z \in I \subseteq \sqrt{I}$, $yz = xz - (x-y)z \in \sqrt{I}$. Now, (xy, xz, yz) contains all monomials with more than one variable, so since $x^n, y^n, z^n \notin I$ for any n, none of them are in $\sqrt{I}$ either, so $\sqrt{I} = (xy, xz, yz)$.

We know that $V(I) = V(\sqrt{I})$ by the Nullstellensatz, but the problem asks to show it directly. $a = (x, y, z) \in V(\sqrt{I})$ iff $xy = xz = yz = 0$ iff at least two of x, y, and z are 0. On the other hand, $a = (x, y, z) \in V(I)$ iff $xy = 0$ and $(x-y)z = 0$ iff either $x = 0$ and $-yz = 0$ or $y = 0$ and $xz = 0$ iff at least two of x, y, and z are 0.

$V(I) = \{(x, 0, 0)\} \cup \{(0, y, 0)\} \cup \{(0, 0, z)\} = V((y, z)) \cup V((x, z)) \cup V((x, y))$, and since none of these varieties is contained in the others, $V(I)$ is reducible. Finally, $\sqrt{I}$ is not prime since $xy \in \sqrt{I}$ but $x, y \notin \sqrt{I}$.

7. **Cox, Little, and O'Shea #8.2.3** *In this exercise, we will study how lines in $\mathbb{R}^n$ relate to points at infinity in $P^n(\mathbb{R})$. We will use the decomposition $P^n(\mathbb{R}) = \mathbb{R}^n \cup P^{n-1}(\mathbb{R})$. Given a line L in $\mathbb{R}^n$, we can parametrize L by the formula $a + bt$, where $a \in L$ and b is a nonzero vector parallel to L. In coordinates, we write this parametrization as $(a_1 + b_1 t, \ldots, a_n + b_n t)$.*

   (a) We can regard L as lying in $P^n(\mathbb{R})$ using the homogeneous coordinates

   $$[1 : a_1 + b_1 t : \cdots a_n + b_n t].$$

   To find out what happens as $t \to \pm\infty$ divide by t to obtain

   $$\left[\frac{1}{t} : \frac{a_1}{t} + b_1 : \cdots \frac{a_n}{t} + b_n\right].$$

   As $t \to \pm\infty$, what point of $H = P^{n-1}(\mathbb{R})$ do you get?
   Solution. As we take the limit, the first coordinate, as well as every term of the form a_i/t goes to 0. So we obtain

   $$\left[\frac{1}{t} : \frac{a_1}{t} + b_1 : \cdots \frac{a_n}{t} + b_n\right] \mapsto [0 : b_1 : \cdots b_n] \mapsto [b_1 : \cdots : b_n] \in H.$$

   (b) The line L will have many parametrizations. Show that the point of $P^{n-1}(\mathbb{R})$ given by part (a) is the same for all parametrizations of L. Hint (from the book): Two nonzero vectors are parallel if and only if one is a scalar multiple of the other.
   Solution. Any parametrization of L has the form $(f_1(t), \ldots, f_n(t))$, where there is a continuous invertible function $g(t) : \mathbb{R} \to \mathbb{R}$ such that $f_i(t) = a_i + b_i g(t)$ for all i. In particular, $\lim_{t\to\infty} g(t) = \pm\infty$. Then we have

   $$\begin{aligned} [1 : f_1(t) : \ldots : f_n(t)] &= [1 : a_1 + b_1 g(t) : \cdots : a_n + b_n g(t)] \\ &= \left[\frac{1}{t} : \frac{a_1}{g(t)} + b_1 : \cdots : \frac{a_n}{g(t)} + b_n\right] \\ &\mapsto [0 : b_1 : \cdots : b_n] \\ &\mapsto [b_1 : \cdots : b_n] \in H. \end{aligned}$$

   (c) Parts (a) and (b) show that a line L in $\mathbb{R}^n$ has a well-defined point at infinity in $H = P^{n-1}(\mathbb{R})$. Show that two lines in $\mathbb{R}^n$ are parallel if and only if they have the same point at infinity.
   Solution. By part (a), the line L parametrized as $a + bt$ maps to the point $[b_1 : \cdots : b_n] \in H$, and by part (b), this is is independent of parametrization. Let L' be another line, parametrized as $a' + b't$. Then by part a, L' maps to the point $[b'_1 : \cdots : b'_n] \in H$, which is the same point L maps to if and only if $b = \pm b'$ i.e. if and only if L and L' are parallel.

8. **Cox, Little, and O'Shea #8.2.7** *In this exercise, we will study when a nonhomogeneous polynomial has a well-defined zero set in* $P^n(k)$*. Let* k *be an infinite field. We will show that if* $f \in k[x_0, ..., x_n]$ *is not homogeneous, but* f *vanishes on all homogeneous coordinates of some* $p \in P^n(k)$*, then each of the homogeneous components* f_i *of* f *(see Definition 6 of Chapter 7, Section 1) must vanish at* p*.*

   (a) Write f as a sum of its homogeneous components $f = \sum_i f_i$. If $p = (a_0, \ldots, a_n)$ then show that

   $$f(\lambda a_0, \ldots, \lambda a_n) = \sum_i f_i(\lambda a_0, \ldots, \lambda a_n) = \sum_i \lambda^i f(a_0, \ldots, a_n)$$

   Solution. The first equality is by definition. For the second equality note that if $h(x_0, \ldots, x_n) = x_0^{e_0} \cdots x_n^{e_n}$, then

   $$h(\lambda x_0, \ldots, \lambda x_n) = (\lambda x_0)^{e_0} \cdots (\lambda x_n)^{e_n} = \lambda^{e_0 + \cdots + e_n} x_0^{e_0} \cdots x_n^{e_n},$$

   in other words, plugging in λx_i for each x_i multiplies any monomial by λ to the power of the degree of that monomial. The same is true for sums of monomials. Let $f_i = \sum_j h_{ij}$, where each h_{ij} is a monomial (of degree i) times a constant. Then,

   $$f_i(\lambda a_0, \ldots, \lambda a_n) = \sum_j h_{ij}(\lambda a_0, \ldots, \lambda a_n) = \lambda^i h_{ij}(a_0, \ldots, a_n) = \lambda^i f_i(a_0, \ldots, a_n).$$

   (b) Deduce that if f vanishes for all nonzero $\lambda \in k$, then $f_i(a_0, \ldots, a_n) = 0$ for all i
   Solution. Let $g_i(\lambda) = f_i(\lambda a_0, \ldots, \lambda a_n)$, and $g(\lambda) = f(\lambda a_0, \ldots, \lambda a_n)$. Note that these are single-variable polynomials, since p is fixed. By part (a), $g(\lambda) = \sum_i g_i(\lambda)$, and also by part (a), $g_i(\lambda) = c_i \lambda^i$ for some $c_i \in k$. If $f(\lambda a_0, \ldots, \lambda a_n) = 0$ for all nonzero $\lambda \in k$, then $g(\lambda) = 0$ for all nonzero $\lambda \in k$. But this means that g is the zero polynomial, which means all its coefficients must vanish, so $g_i = 0$ for all i, and therefore $f_i(\lambda a_0, \ldots, \lambda a_n)$ for all i and all λ.