Field Extensions (cont.)

Prop: An extension field K of F is a vector space over F

Pf: check axioms

The <u>degree</u> $[K:F] := \dim_F K$

Examples:

a) $\mathbb{C}/\mathbb{R}$: $\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$, so

$S = \{1, i\}$, $[\mathbb{C}:\mathbb{R}] = 2$

b) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$: $\mathbb{Q}(\sqrt{2}) = \{a+b\sqrt{2} \mid a,b \in \mathbb{Q}\}$

since $\frac{a+b\sqrt{2}}{c+d\sqrt{2}} = \frac{(a+b\sqrt{2})(c-d\sqrt{2})}{c^2-2d^2}$

so $S = \{1, \sqrt{2}\}$ $[\mathbb{Q}(\sqrt{2}):\mathbb{Q}] = 2$

c) $\mathbb{F}_p(x)/\mathbb{F}_p$: $1, x, x^2, \dots$ are linearly indep.,

so $[\mathbb{F}_p(x):\mathbb{F}_p] = \infty$

Goal: form field extensions by adding roots of polys.

F: field, $p(x) \in F[x]$ irred., nonconstant

Let $K := F[x]/(p(x))$

Prop: K is a field

Pf: $p(x)$ irred. $\Rightarrow$ $p(x)$ prime (since $F[x]$ is a PID)

$\Rightarrow (p(x))$ prime

$\Rightarrow (p(x))$ maximal (since $F[x]$ is a PID)

$\Rightarrow K$ is a field. $\square$

Thm: K is an extension field of F containing a root θ of p. If $\deg p = n$, then $\{1, \theta, \ldots, \theta^{n-1}\}$ is a basis for K over F, so $[K:F] = n$.

Pf: $F \overset{}{\underset{\text{inclusion}}{\hookrightarrow}} F[x] \underset{\text{projection}}{\twoheadrightarrow} F[x]/(p) = K$,

and the composition of these maps is inj.,

so $F \subseteq K$.

Let $\theta = x + (p(x)) \in F[x]/_{(p(x))} = K$

Then,

$$p(\theta) = p\big(x + (p(x))\big) \overset{\text{proj. is hom.}}{=} p(x) + (p(x)) = 0 + (p(x)),$$

which is 0 in K.

Let $a(x) \in F[x]$. Since $F[x]$: Euc. dom.,

$$a(x) = q(x)\,p(x) + r(x), \quad \deg r < n.$$

So $\bar{a} = r + (p) \in K$, so K is spanned by $1, \theta, \dots, \theta^{n-1}$. On the other hand, if $1, \dots, \theta^{n-1}$ are linearly dep., then $\exists b_0, \dots, b_{n-1} \in F$ not all 0 s.t. $b_0 + b_1\theta + \dots + b_{n-1}\theta^{n-1} = 0 \in K$.

Thus,

$$b_0 + b_1 x + \cdots + b_{n-1} x^{n-1} + (p(x)) = 0 + (p(x)) \text{ in } K,$$

so $b_0 + b_1 x + \cdots + b_{n-1} x^{n-1}$ is a multiple of $p(x)$ in $F[x]$. But this is impossible since $\deg p = n > n-1$. $\square$

Remark: need p to be <u>irred.</u>, otherwise K is not a field

Trick to reduce polys. mod p.

$p(x) = x^n + p_{n-1} x^{n-1} + \cdots + p_1 \theta + p_0$

$p(\theta) = 0$, so

$$\theta^n = -(p_{n-1}\theta^{n-1} + \cdots + p_1 \theta + p_0)$$

$$\begin{aligned} \theta^{n+1} = \theta\theta^n &= -(p_{n-1}\theta^n + \cdots + p_1\theta^2 + p_0\theta) \\ &= -p_{n-1}\left(-(p_{n-1}\theta^{n-1} + \cdots + p_1\theta + p_0)\right) \\ &\quad + \cdots + p_1\theta^2 + p_0\theta) \quad \text{etc.} \end{aligned}$$

Example: $F = \mathbb{R}$, $p(x) = x^2+1$

$$K = \mathbb{R}[x]/(x^2+1) = \{a+b\theta \mid a,b \in \mathbb{R}\} \qquad \theta^2 = -1$$

since $\theta^2+1=0$

$$(a+b\theta)(c+d\theta) = (ac-bd) + (ad-bc)\theta$$

So $K \cong \mathbb{C}$!

Two isoms.: $\theta \mapsto \pm i$

Many more examples in D&F (p. 515-516)

Let's relate our new construction w/ a more "intuitive" way of thinking about field ext'ns

Def: Let $F \subseteq K$, $\alpha, \beta, \ldots \in K$.

$F(\alpha, \beta, \ldots)$ is the smallest subfield of K containing F and $\alpha, \beta, \ldots$

Equivalently, $F(\alpha, \beta, \ldots)$ = intersection of all subfields of K w/ this property

Simple extn: $E = F(\alpha)$ ← primitive elt.

Examples:

a) $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \overset{\text{nontriv.}}{=} \mathbb{Q}(\sqrt{2} + \sqrt{3})$ is simple

b) $\mathbb{Q}(\sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2}, \ldots)$ is not simple

Thm: $p(x) \in F[x]$: irred.

Let K: extn field of F containing a root α of p.

Then, $F[x]/(p(x)) \cong F(\alpha) \subseteq K$

Pf: Consider the map given by $x + (p) \overset{\varphi}{\longmapsto} \alpha$ i.e.

$g(x) + (p(x)) \longmapsto g(a)$.

- Well defined: $g(a) = 0$ if $g \in (p)$
- Ring homom.: check the axioms
- Injective: $\ker \varphi$ is an ideal, which for a field is either (0) or $F[x]/(p)$. Not the latter since $1 \mapsto 1$

- Surjective: image is a field containing F and α

$\square$

Cor: Let $E = F(\alpha) \subseteq K$ w/ $[K:F] = n < \infty$. Then,

a) $\exists$ irred. $p(x) \in F[x]$ s.t. $p(\alpha) = 0$.

b) $\deg p = n$

c) $E \cong F[x]/(p)$

d) E is indep. of the choice of root of p

i.e. if $p(\beta) = 0$, $F(\alpha) \cong F(\beta)$.

Pf: Since $[K:F] = n$, $1, \alpha, \dots, \alpha^n$ are linearly dep. i.e.

$$a_n \alpha^n + \cdots + a_1 \alpha + a_0 = 0$$

Let $p(x)$ be an irred. factor of $a_n x^n + \cdots + a_1 x + a_0$

b) This follows from our first theorem today

c) Follows from previous theorem

d) Follows from c)

Extension Theorem: Let $\varphi : F \xrightarrow{\sim} F'$ be an isom. of fields. Let $p(x) \in F[x]$ be irred., and let $p'(x) \in F'[x]$ be the irred. poly obtained by applying φ to the coeffs. of p.

Let α be a root of p (in some extn of F)

Let β be a root of p' (in some extn of F')

Then $\exists$ isom.

$$\sigma : F(\alpha) \xrightarrow{\sim} F'(\beta)$$

$$f \longmapsto \varphi(f) \qquad (\sigma|_F = \varphi)$$

$$\alpha \longmapsto \beta$$

(Seems unintuitive now, but useful later)

Pf (skip in class): Let $\tilde{\varphi}$ be the isom.

$$\tilde{\varphi} : F[x] \xrightarrow{\sim} F'[x]$$

$$f \mapsto \varphi(f)$$

$$x \mapsto x$$

Then $\tilde{\varphi}$ maps $(p(x))$ to $(p'(x))$, so it induces an isom

$$F[x]/_{(p(x))} \xrightarrow{\sim} F'[x]/_{(p'(x))}$$

$$f \overset{\sim}{\longmapsto} \varphi(f) + (p')$$

$$x + (p) \overset{\sim}{\longmapsto} x + (p')$$

Combining this w/ our previous isoms., σ is the map

$$\begin{array}{ccccccc}
F(\alpha) & \xrightarrow{\sim} & F[x]/_{(p(x))} & \xrightarrow{\sim} & F'[x]/_{(p'(x))} & \xrightarrow{\sim} & F'(\beta) \\
f & \longmapsto & f + (p) & \longmapsto & \varphi(f) + (p') & \longmapsto & \varphi(f) \\
\alpha & \longmapsto & x + (p) & \longmapsto & x + (p') & \longmapsto & \beta
\end{array}$$

$\square$

$$\begin{array}{ccccc}
\sigma : & F(\alpha) & \xrightarrow{\sim} & F'(\beta) \\
 & | & & | \\
\varphi : & F & \xrightarrow{\sim} & F'
\end{array}$$