Announcements:

Final exam: Tues. 5/7 8:00am-11:00am,
1047 Sidney Lu Mech. E. Bldg.

Exam will be cumulative

See policy email sent last night for full details

- TWO reference sheets allowed
- practice problems (sol'ns by the weekend)
- regrade requests will only span a couple days

Review session: Sunday 5/5 11:00pm-1:00pm,
Digital Computer Lab. 1310
(come w/ questions)

Office hours: Friday and Monday 2:00pm-3:00pm
or by email/appointment

ICES questionnaires: go.illinois.edu/ices-online

Final exam review

(See previous review topics)

Integral domains, poly. rings, irreducibility

Basic tools: irreducibility, field ext'ns, degrees, splitting fields, min'l polys., tower law

Constructibility: 4 classical problems, type of ext'ns allowed

Separability: derivative criterion, irreds. over char 0 or fin. field

Galois theory:

- Compute Galois gps. (both up to isom. class and via generators and rel'ns)
- Galois correspondence (draw diagrams etc.)
- Solvability by radicals
- Examples: cyclotomic ext'ns, finite fields, cubics, composite ext'ns

Algebraic geometry:

- Ideals, varieties, basic properties
- Radical ideals, Nullstellensatz (all forms)

Noetherian rings

Prime $\leftrightarrow$ irred., max'l $\leftrightarrow$ pt.

Coordinate ring

Projective space (all def'ns)

Homogeneous ideals, projective varieties

Projective Nullstellensatz

Specific examples

- for studying, look at lecture notes, homework/midterm problems, practice problems, textbook
- midterm length < final exam length < $2 \cdot$ midterm length
- understand how topics mesh (e.g. ED/PID/UFD w/ alg. geom.)
- understand theory and examples

Example problems:

1) a) Prove that $V = \{(a, a^2, a^3) \mid a \in k\}$ is an irreducible affine variety.

Pf: $V = V(I)$ for $I = (x^2 - y, x^3 - z)$, so V is a variety. We show V is irred. by showing that I is prime. Can show this using the def'n of prime: if $f \cdot g \in I$, f or $g \in I$. Alternatively,

$$k[x,y,z]/I \cong k[x]$$

$$x \longmapsto x$$
$$y \longmapsto x^2$$
$$z \longmapsto x^3$$

and since $k[x]$ is an int. domain, I is prime.

b) Prove that $W = \{[b^3 : ab^2 : a^2b : a^3] \mid a, b \in \mathbb{C}$ not both $0\}$ is an irred proj. variety.

Pf: $W = V(J)$ where $J = (xw - yz,\ xz - y^2,\ yw - z^2)$ (fill in the details). J is a homog. ideal, So W is a proj. variety. J is prime since

$$\left.\begin{aligned} k[x,y,z,w]/J &\cong k[s,t] \\ x &\longmapsto s^3 \\ z &\longmapsto s^2t \\ y &\longmapsto st^2 \\ w &\longmapsto t^3 \end{aligned}\right\} \text{needs more details}$$

So W is irred. $\square$

2) Compute the Galois gp. / Galois corresp. for $f(x) = (x^3+x+1)(x^3+1)$ over $\mathbb{F}_2$

Sol'n: $x^3+1 = (x+1)(x^2-x+1)$ (over any field)

Over $\mathbb{F}_2$, x^3+x+1 is irred. (no root)

So $f(x) = (x+1)(x^2-x+1)(x^3+x+1) \in \mathbb{F}_2[x]$.

The largest irred. factor has deg 3, so

$Sp_{\mathbb{F}_2} f = \mathbb{F}_{2^3}$ (D&F Prop 18)

We have $G := \mathrm{Gal}(f) = \mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2) = \mathbb{Z}/3\mathbb{Z}$

$$\begin{array}{ccc} I & & \mathbb{F}_8 \\ | & & | \\ G & & \mathbb{F}_2 \end{array}$$

G abelian, hence everything normal / Galois

3) Prove that a quotient of a PID R by a prime ideal I is again a PID.

Pf: If $I = (0)$, then $R/I \cong R$ is a PID

If $I \neq (0)$, then I is maximal (D&F Prop 8.7), so R/I is a field, hence a PID. $\square$

4) Let K/F be a nontriv. Galois ext'n of odd order, and let $\alpha \in K \setminus F$. Prove that

$$|\{\sigma \in \mathrm{Gal}(K/F) \mid \sigma(\alpha) \neq \alpha\}| > |\{\sigma \in \mathrm{Gal}(K/F) \mid \sigma(\alpha) = \alpha\}|$$

Pf: Since K/F is Galois, $\mathrm{Gal}(K/F(\alpha))$ is a proper subgp. of $\mathrm{Gal}(K/F)$. Since $[K:F]$ is odd, so is

$|\mathrm{Gal}(K/F)|$, so every proper subgp. has index ≥ 3.

Therefore, the subset of $\mathrm{Gal}(K/F)$ of automs. that fix α is $\leq \frac{1}{3}$ of the total. $\square$

b) Give a nontrivial ext'n of odd order s.t.

$$|\{\sigma \in \mathrm{Aut}(K/F) \mid \sigma(\alpha) \neq \alpha\}| \leq |\{\sigma \in \mathrm{Aut}(K/F) \mid \sigma(\alpha) = \alpha\}|$$

Sol'n: Let $F = \mathbb{Q}$, $K = \mathbb{Q}(\sqrt[3]{2})$

$\mathrm{Aut}(K/F) = 1$, so id is the only elt., and this fixes $\sqrt[3]{2} \in K \setminus F$. $\square$