Recall: k: alg. closed field

$$V(I) := \{a \in k^n \mid f(a) = 0 \ \forall f \in V\}$$

$$I(V) = \{f \in k[x_1, .., x_n] \mid f(a) = 0 \ \forall a \in V\}$$

$$\sqrt{I} = \{f \in k[x_1, .., x_n] \mid f^n \in I \text{ for some } n \geq 0\}$$

Nullstellensatz (strong form): $I(V(I)) = \sqrt{I}$

Moreover, we have inverse bijections

$$\begin{tikzcd}
\begin{matrix}\text{alg. varieties} \\ V \subseteq k^n\end{matrix} \arrow[r, shift left, "I"] & \begin{matrix}\text{radical ideals} \\ I \subseteq k[x_1, .., x_n]\end{matrix} \arrow[l, shift left, "V"]
\end{tikzcd}$$

Weak form:

Let $I \subseteq k[x_1, .., x_n]$ be an ideal. Then $V(I) = \emptyset$ if and only if $1 \in I$ (and so $I = k[x_1, .., x_n]$)

Moreover, we have inverse bijections

$$\begin{tikzcd}
\begin{matrix}\text{points} \\ a \in k^n\end{matrix} \arrow[r, shift left, "I"] & \begin{matrix}\text{max'l ideals} \\ I \subseteq k[x_1, .., x_n]\end{matrix} \arrow[l, shift left, "V"]
\end{tikzcd}$$

[Last time: $\begin{tikzcd}
\begin{matrix}\text{irred.} \\ \text{vars.}\end{matrix} \arrow[r, shift left, "I"] & \begin{matrix}\text{prime ideals} \\ I \subseteq k[x_1, .., x_n]\end{matrix} \arrow[l, shift left, "V"]
\end{tikzcd}$]

Lemma:

a) $I(a) = (x_1 - a_1, \ldots, x_n - a_n)$

b) $I(a)$ is maximal

Pf: $J := (x_1 - a_1, \ldots, x_n - a_n) \subseteq I(a)$, so we'll prove that J is max'l. Under the quotient map $k[x_1, \ldots, x_n] \to k[\ldots]/J$, $p(x) \mapsto p(a)$, so $k[x_1, \ldots, x_n]/J \cong k$, a field, so $J = I(a)$

$\square$

Prop: Every max'l ideal is of the form $I(a)$ for some $a \in k^n$

Pf when k is uncountable (e.g. $\mathbb{C}$, <u>not</u> $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_p}$):

Let $I \subseteq k[x_1, \ldots, x_n]$ be a max'l ideal, and let $F = k[x_1, \ldots, x_n]/I$. $k \subseteq F$ since $k \cap I = 0$, so either $F = k$ or F is a transcendental ext'n of k. In the former case, $I = I(a) = I((a_1, \ldots, a_n))$ where $x_i \mapsto a_i$. In the latter case, $\dim_k F$ is at most countable since $\dim_k k[x_1, \ldots, x_n]$ is countable, and the quotient map is a vector space homom. On the other hand, let $t \in F$ be trans. / k. Now,

$\left\{ \frac{1}{t-a} \middle| a \in k \right\}$ is an uncountable linearly indep. set,

a contradiction. $\square$

(Pf: If $\frac{c_1}{t-a_1} + \cdots + \frac{c_n}{t-a_n} = 0$, then

$c_1(t-a_2)\cdots(t-a_n) + \cdots + c_n(t-a_1)\cdots(t-a_{n-1}) = 0$,

and setting $t=a_i$ shows that each $c_i = 0$)

Pf of weak Nullstellensatz: Every proper ideal I is contained in a max'l ideal $I(a)$ (don't need Zorn's lemma since ring is Noetherian). If $V(I)=\phi$, then $V(I(a)) = \phi$, but this contradicts the fact that $V(I(a)) = \{a\}$. $\square$

Pf of strong Nullstellensatz: Already proved $\sqrt{I} \subseteq I(V(I))$ (lecture 36).

Since $k[x_1,\dots,x_n]$ is Noetherian, I is finitely-generated i.e. $I = (f_1,\dots,f_m)$. Let $g \in I(V(I))$. Introduce a new variable x_{n+1}, and consider

$$I' = (f_1,\dots,f_n, x_{n+1}g - 1) \subseteq k[x_1,\dots,x_{n+1}]$$

For any $a \in k^{n+1}$, if $f_1(a) = \cdots = f_n(a) = 0$, then also $g(a) = 0$ (since $g \in I(V(I))$), so $x^{n+1}g - 1 \neq 0$. Thus, $V(I') = \emptyset$.

By the weak form of the Nullstellensatz, $1 \in I'$, so

$$1 = h_1 f_1 + \cdots + h_m f_m + h_{m+1}(x_{n+1}g - 1) \text{ for some } h_i \in k[x_1, \ldots, x_{n+1}]$$

Let $y = x_{n+1}^{-1}$, and multiply by y^N, $N \gg 0$:

$$y^N = p_1 f_1 + \cdots + p_m f_m + p_{m+1}(g - y) \text{ for some } p_i \in k[x_1, \ldots, x_n, y]$$

Plug in $y = g$:

$$g^N = \tilde{p}_1 f_1 + \cdots + \tilde{p}_m f_m \in I \subseteq k[x_1, \ldots, x_n]$$

where $\tilde{p}_i(x_1, \ldots, x_n) = p_1(x_1, \ldots, x_n, g) \in k[x_1, \ldots, x_n]$

So $g \in \sqrt{I}$, and so $I(V(I)) = \sqrt{I}$.

$\square$

If time: coordinate ring

Def: The coord. ring of a variety V is

$$k[V] = \{ f : V \to k \mid f = g|_V \text{ for some } g \in k[x_1, \dots, x_n] \}$$

i.e. $k[V] = k[x_1, \dots, x_n] / I(V)$

since $I(V)$ is the kernel of the ring homom.

$$k[x_1, \dots, x_n] \to k[V]$$

Cor of previous results:

a) V irred $\Leftrightarrow$ $k[V]$ int. domain

b) V pt. $\Leftrightarrow$ $k[V] \cong k$

c) For any variety V,

$$\{\text{pts in } V\} \xleftrightarrow{\text{bij.}} \{\text{max'l ideals in } k[V]\}$$

Pf: a) V irred. $\Leftrightarrow$ $I(V)$ prime $\Leftrightarrow$ $\underbrace{k[V] = k[x_1,\dots,x_n]/I}_{\text{int. domain}}$

b) V pt. $\Leftrightarrow$ $I(V)$ maximal $\Leftrightarrow$ $\underbrace{k[V] = k[x_1,\dots,x_n]/I}_{\text{field}}$

(and this field $\cong k$ since we know $I = I(a)$ for some $a \in k$)

c) By the 4th ring isom. thm., for a ring R and ideal I,

$$\{\text{ideals in } R \text{ containing } I\} \overset{\text{bij}}{\longleftrightarrow} \{\text{ideals in } R/I\}$$

Maximality is preseved under this bijection,

so combine this w/ the weak Nullstellensatz $\square$