Announcements:

Final exam: Tues. 5/7 8:00am - 11:00am,
1047 Sidney Lu Mech. E. Bldg.
(email ASAP w/ any issues)
Exam will be cumulative

Problem session tomorrow: I need to leave ~15 mins. early

Recall: k: alg. closed field

$$V(I) := \{a \in k^n \mid f(a) = 0 \ \forall f \in V\}$$

$$I(V) = \{f \in k[x_1, \dots, x_n] \mid f(a) = 0 \ \forall a \in V\}$$

$$\sqrt{I} = \{f \in k[x_1, \dots, x_n] \mid f^n \in I \text{ for some } n \geq 0\}$$

Hilbert's Nullstellensatz (strong form): $I(V(I)) = \sqrt{I}$.

Moreover, we have inverse bijections

$$\begin{array}{c} \text{alg. varieties} \\ V \subseteq k^n \end{array} \underset{V}{\overset{I}{\rightleftarrows}} \begin{array}{c} \text{radical ideals} \\ I \subseteq k[x_1, \dots, x_n] \end{array}$$

Hilbert's Nullstellensatz (weak form):

Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. Then $V(I) = \emptyset$ if and only if $1 \in I$ (and so $I = k[x_1, \dots, x_n]$)

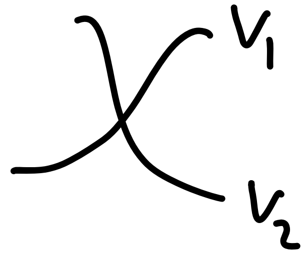
Prime ideals are radical since in a prime ideal I,

$ab \in I \Rightarrow a \in I$ or $b \in I$, so $a^n \in I \Rightarrow a \in I$

Def: A variety V is irreducible if whenever

$V = V_1 \cup V_2$ for varieties V_1 and V_2, $V = V_1$ or $V = V_2$.

Prop: V irred $\Leftrightarrow$ $I := I(V)$ prime



Pf: $\Rightarrow$) Let $f_1 f_2 \in I$

Let $V_i = V \cap V(f_i) = V(I + (f_i))$

$= \{a \in V \text{ s.t. } f_i(a) = 0\}$ $(i = 1,2)$

Let $a \in V$. Then $f_1(a) \cdot f_2(a) = f_1 f_2(a) = 0$, so

$f_1(a) = 0$ or $f_2(a) = 0$, and so $V = V_1 \cup V_2$.

Since V irred, $V = V_j$ for $j = 1$ or 2, so

$f_j(a) = 0$ for all $a \in V$, which means that $f_j \in I$,

so I is prime.

$\Leftarrow$) Let $V = V_1 \cup V_2$, and assume $V_1 \subsetneq V$.

This means that $I(V) \subsetneq I(V_1)$ since otherwise

$V = V(I(V)) = V(I(V_1)) = V_1$.

Let $f_1 \in I(V_1) \setminus I(V)$, $f_2 \in I(V_2)$.

Then $f_1 f_2 \in I(V)$ since one of f_1, f_2 is 0 on every point in V.

Since $I(V)$ is prime, must have $f_2 \in I$ (can't have $f_1 \in I$), so $I(V_2) \subseteq I(V)$, so $V_2 \subseteq V \subseteq V_2$, so $V = V_2$ and V irred.

$\square$

Prop: Any variety $V \subseteq k^n$ is a finite union of irred. varieties.

Def: A ring R is <u>Noetherian</u> if every strictly increasing chain of ideals is finite i.e. if

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

then $\exists m$ s.t. $I_k = I_m \ \forall k \geq m$

(sometimes called the ascending chain condition)

Hilbert's Basis Thm: $k[x_1, \dots, x_n]$ is Noetherian

(Pf: D&F Section 9.6, Cor 9.22, uses "leading coeffs.")

Pf of prop: Suppose otherwise. Since V red.,

$$V = \underbrace{V_1 \cup W_1}_{\substack{\text{varieties} \\ V_1, W_1 \subsetneq V}}$$

One of V_1, W_1 must be reducible, say $V_1 = V_2 \cup W_2$, $V_2, W_2 \subsetneq V_1$. Continuing in this manner, we have

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots$$

and letting $I_i = I(V_i)$, we get

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$$

since $V(I_i) = V_i \supsetneq V_{i+1} = V(I_{i+1})$

Since $k[x_1, \dots, x_n]$ is Noetherian, this is impossible. $\square$

What about maximal ideals?

max'l ideals $\subseteq$ prime ideals $\leftrightarrow$ irred. varieties

For $a \in k^n$, let $I(a) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0\} = I(\{a\})$

Lemma:

a) $I(a) = (x_1 - a_1, \ldots, x_n - a_n)$

b) $I(a)$ is maximal

Pf: $J := (x_1 - a_1, \ldots, x_n - a_n) \subseteq I(a)$, so we'll prove that J is max'l. $J = \ker(f \mapsto f(a))$, so $k[x_1, \ldots, x_n]/J \cong \mathrm{im}(f \mapsto f(a)) = k$, a field, so $J = I(a)$ is max'l.

$\square$

Prop: Every max'l ideal is of the form $I(a)$ for some $a \in k^n$

Pf when k is uncountable (e.g. $\mathbb{C}$, <u>not</u> $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_p}$):

Let $I \subseteq k[x_1, \ldots, x_n]$ be a max'l ideal, and let $F = k[x_1, \ldots, x_n]/I$. $k \subseteq F$ since $k \cap I = 0$, so either $F = k$ or F is a transcendental ext'n of k. In the former case, $I = I(a) = I((a_1, \ldots, a_n))$ where $x_i \mapsto a_i$. In the latter case, $\dim_k F$ is at most countable since $\dim_k k[x_1, \ldots, x_n]$ is countable, and the quotient map is a vector space homom. On the other hand, let $t \in F$ be trans./k. Now,

$\left\{ \frac{1}{t-a} \,\middle|\, a \in k \right\}$ is an uncountable linearly indep. set, a contradiction. $\square$

(Pf: If $\frac{c_1}{t-a_1} + \cdots + \frac{c_n}{t-a_n} = 0$, then

$c_1(t-a_2)\cdots(t-a_n) + \cdots + c_n(t-a_1)\cdots(t-a_{n-1}) = 0$,

and setting $t = a_i$ shows that each $c_i = 0$)

Pf of weak Nullstellensatz: Every proper ideal I is contained in a max'l ideal $I(a)$ (don't need Zorn's lemma since ring is Noetherian). If $V(I) = \emptyset$, then $V(I(a)) = \emptyset$, but this contradicts the fact that $V(I(a)) = \{a\}$. $\square$