Recall:

Hilbert's Nullstellensatz (weak form, first version):

Let $f_1(x_1,\dots,x_n), \dots, f_m(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$

Then the system of equations

$$f_1(x_1,\dots,x_n) = \dots = f_m(x_1,\dots,x_n) = 0$$

has <u>no</u> solution in $\mathbb{C}^n$ if and only if $\exists g_1,\dots,g_m \in \mathbb{C}[x_1,\dots,x_n]$ s.t. $f_1g_1+\dots+f_mg_m = 1 \in \mathbb{C}[x_1,\dots,x_n]$

Unless otherwise stated, let k be an alg. closed field

Def: An (affine) <u>algebraic variety</u> (or <u>algebraic set</u>) is a subset $V \subseteq k^n$ of the form

$$V = V(I) := \{a \in k^n \mid f(a) = 0 \ \forall f \in V\}$$

for some subset/ideal $I \subseteq k[x_1,\dots,x_n]$

Prop: I, J: ideals

a) $I \subseteq J \Rightarrow V(I) \supseteq V(J)$

b) $V(I) \cap V(J) = V(I \cup J) = V(I+J)$

c) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$

d) $V(0) = k^n$ and $V(\langle 1 \rangle) = \emptyset$

Def: V: alg. variety. Then set

$$I(V) = \{f \in k[x_1, .., x_n] \mid f(a) = 0 \ \forall a \in V\}$$

$= (a_1, .., a_n)$

Prop: U, V: varieties

a) $U \subseteq V \Rightarrow I(U) \supseteq I(V)$

b) $I(U \cup V) = I(U) \cap I(V)$

c) $I(U \cap V) \supseteq I(U) + I(V)$

Prop:

a) $V = V(I(V))$

b) $I \subseteq I(V(I))$

Pf of a): If $a \in V$, then $\forall f \in I(V)$, $f(a) = 0$, so $a \in V(I(V))$.
Since V is a variety, $V = V(J)$ for some ideal J.
We must have $J \subseteq I(V)$, but then $V(J) \supseteq V(I(V))$, so
$V(I(V)) = V(J) = V$. ∎

i.e. a) is an equality because we already know that every variety V is of the form $V = V(J)$. If we know that $I = I(U)$, then $I(V(I)) = I$ by the same argument.

Hilbert's Nullstellensatz (strong form): $I(V(I)) = \sqrt{I}$.

Moreover, we have inverse bijections

$$\begin{array}{ccc} \text{alg. varieties} & \xrightarrow{I} & \text{radical ideals} \\ V \subseteq k^n & \xleftarrow[V]{} & I \subseteq k[x_1, \ldots, x_n] \end{array}$$

Pf of easy direction: If $f \in \sqrt{I}$ then $f^n \in I$ for some n. If $a \in V(I)$, then

$0 = f^n(a) = (f(a))^n$, so $f(a) = 0$ since

$k[x_1, \ldots, x_n]$ is an int. domain.

$\square$

Cor: Hilbert's Nullstellensatz (weak form, second version)

Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then $V(I) = \emptyset$ if and only if $1 \in I$ (and so $I = k[x_1, \ldots, x_n]$)

Pf: By the strong form,

$$\sqrt{I} = I(V(I)) = I(\emptyset) = k[x_1, \dots, x_n],$$

So $1 \in \sqrt{I}$. This means that $1^n \in I$ for some n,

so $1 = 1^n \in I$ $\square$

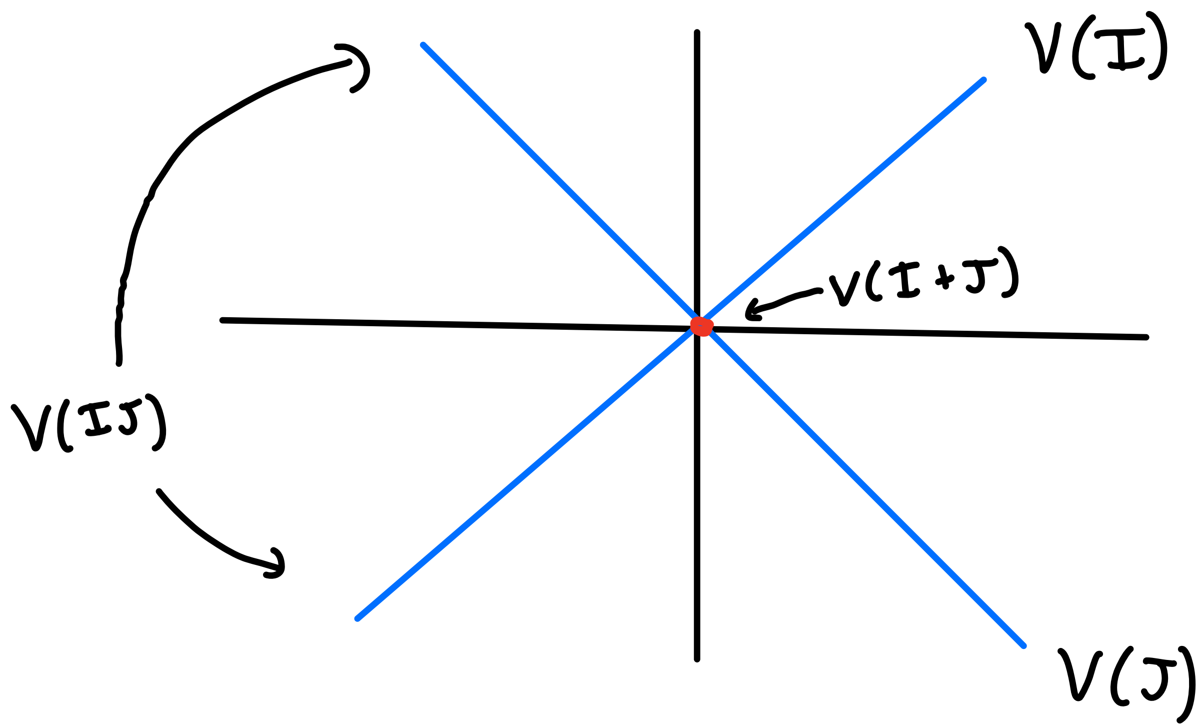
(in practice, the weak form is used to prove the strong form)

Examples:

a) $k = \mathbb{C}$ (or $\mathbb{R}$), $n = 2$

$I = (x-y)$, $\quad J = (x+y) \quad I + J = (x, y)$

$I \cap J = IJ = ((x-y)(x+y))$



$$I(V(J)) = \{ f \in \mathbb{C}[x,y] \mid f(x,-x) = 0 \ \forall x \}$$

If $(x+y) \mid f(x,y)$, (recall: $k[x_1, \dots, x_n]$ is a UFD)

then $f(x,-x) = 0$

So $J \subseteq I(V(J))$. Can it be bigger?

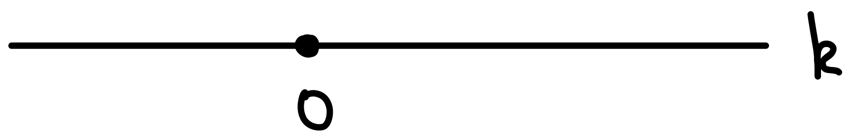
Yes, but in this case $I(V(J)) = J$

$$I(V(I+J)) = \{ f \in k[x,y] \mid f(0,0) = 0 \}$$

$=$ all functions w/out a constant term

$= (x,y) = I + J$

b) $n = 1$ $\quad I = (x^2) \subseteq k[x]$



$V(I) = 0$, but $I(V(I)) = (x) \supsetneq I$

$= \sqrt{I}$