Midterm 3: tomorrow 7:00-8:30 Loomis Lab. 144

Topics: everything through Galois theory

Practice problems + policies: see email

Problem session tomorrow 10am-12pm

3rd floor of Altgeld (345 or 347)

Midterm 3 review

Integral domains, poly. rings, irreducibility

Basic tools: irreducibility, field ext'ns, degrees, splitting fields, min'l polys., tower law

Constructibility: 4 classical problems, type of ext'ns allowed

Separability: derivative criterion, irreds. over char 0 or fin. field

Galois theory:

- Compute automorphisms, fixed fields
- Characterization of Galois ext'n (autom. gp. size, poly. splitting)
- Galois correspondence (inc. properties e.g. normal subgps.)

- trace, norm, and sym. funs. (lie in base field)

Important cases:

- finite fields
- cyclotomic ext'ns

Compute Galois gps.

- discriminant (def and A_n criterion)
- compute Gal. gp. for deg 2,3
- gens. and rel'ns and/or cycle type
  (find some automs. and determine the gp. they gen.)

Solvability by radicals:

- Solvable gps and solvability criterion (Galois' thm)
- Cardano's formula (don't need to memorize)
- Prove that a poly. is/isn't solvable by radicals

Practice probles (pf. sketches posted on website)

14.2.10) Determine the Galois gp of x^8-3 over $\mathbb{Q}$.

$$K := Sp_{\mathbb{Q}}\, x^8-3 = \mathbb{Q}(\zeta_8, \sqrt[8]{3}) = \mathbb{Q}(i, \sqrt{2}, \sqrt[8]{3})$$

$[K:\mathbb{Q}] = 32$, so since $K/\mathbb{Q}$ is Galois,

$$|\underbrace{\mathrm{Gal}(K/\mathbb{Q})}_{G}| = 32. \quad G = \{\sigma_{\pm\pm a}\}$$

$$\sigma_{++a} : \begin{cases} i \mapsto i \\ \sqrt{2} \mapsto \sqrt{2} \\ \sqrt[8]{3} \mapsto \zeta^a \sqrt[8]{3} \end{cases} \qquad \sigma_{+-a} : \begin{cases} i \mapsto i \\ \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt[8]{3} \mapsto \zeta^a \sqrt[8]{3} \end{cases}$$

$$\sigma_{-+a} : \begin{cases} i \mapsto -i \\ \sqrt{2} \mapsto \sqrt{2} \\ \sqrt[8]{3} \mapsto \zeta^a \sqrt[8]{3} \end{cases} \qquad \sigma_{--a} : \begin{cases} i \mapsto -i \\ \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt[8]{3} \mapsto \zeta^a \sqrt[8]{3} \end{cases}$$

$$\sigma = \sigma_{++1} \qquad \tau = \sigma_{-+0} \qquad \rho = \sigma_{+-0}$$

$$\mathrm{Aut}(K/\mathbb{Q}(\sqrt[3]{8})) = \langle \tau, \rho \rangle \cong V_4$$

$$\mathrm{Aut}(K/\mathbb{Q}(\zeta)) = \langle \sigma \rangle \cong C_8$$

$$\tau\sigma : \begin{cases} i \mapsto -i \\ \sqrt{2} \mapsto \sqrt{2} \\ \sqrt[8]{3} \mapsto \rho \sqrt[8]{3} \mapsto \rho^7 \sqrt[8]{3} \end{cases} \qquad = \sigma^{-1}\tau$$

$$\rho\sigma : \begin{cases} i \mapsto i \\ \sqrt{2} \mapsto -\sqrt{2} \\ \sqrt[8]{3} \mapsto \rho^3 \sqrt[8]{3} \end{cases} \qquad = \sigma^3\tau$$

So C_8 is normal i.e.

$$G = C_8 \rtimes V_4$$

Determine the Galois gp. of x^3+2x+2 over $\mathbb{Q}$

Irred. by Eis. $(p=2)$

So $G := \mathrm{Gal}(x^3+2x+2) = A_3$ or S_3

$D = -4(2^3) - 27(2^2) < 0$ is not a $\square$ in $\mathbb{Q}$

So $G = S_3$

14.4.4) Let $f(x) \in F[x]$ be an irred. poly. of deg n over F. Let $L = Sp_F f$, and let α be a root of f in L. If K is any Galois ext'n of F, show that

$$f(x) = \underbrace{p_1(x) \cdots p_m(x)}_{\text{irred. of deg } d} \in K[x]$$

where $d = [K(\alpha) : K] = [(L \cap K)(\alpha) : L \cap K]$ and $m = n/d = [F(\alpha) \cap K : F]$

Pf: Every factor of f lies in $L[x]$, so the irred. factorization of f in $K[x]$ equals the irred. factorization of f in $(K \cap L)[x]$, so the two def'ns of d are the same. We also have

$$n = [F(\alpha) : F] = \underbrace{[F(\alpha) : F(\alpha) \cap K]}_{d} \underbrace{[F(\alpha) \cap K : F]}_{m}, \text{ so}$$

the def of m is consistent too.

Let $H \leq \mathrm{Gal}(L/F)$ correspond to the int field $L \cap K$. By our construction of min'l polys, for any root α of f in L,

$$m_{\alpha, L\cap K}(x) = \prod_{\beta \in H\alpha} (x - \beta)$$

Thus, the degs. of the irred. factors of $f(x)$ over $L \cap K$ equal the sizes of the H-orbits of $S := \{\text{roots of } f\}$

Since K/F is Galois, by prop. 4 of the Fun. Thm., $H \trianglelefteq G$. By Dummit & Foote Ex. 4.9a, since G acts transitively on S and H is normal, the H-orbits must be the same size.

(pf: transitivity $\Rightarrow S = \{g\alpha \mid g \in G\}$. Orbits are $Hg\alpha = gH\alpha$, which has order $|H\alpha|$)

Thus, all the p_i have the same degree, and this degree equals

$$\deg m_{\alpha, L\cap K}(x) = [(L\cap K)(\alpha) : L\cap K]$$

$\square$