HW9 posted (due Wed. 4/24)

Assume char $F = 0$ (or just let it not divide anything we don't want it to)

Thm (ancients, Cardano, Ferrari): All deg ≤ 4 polys are solvable

Thm (Abel-Ruffini): There is no general formula by radicals for $f_{gen}^{(n)}$, $n \geq 5$.

Thm (Galois):

a) $f(x)$ is solvable by radicals $\Longleftrightarrow$ Gal f is a "solvable gp"

b) $\exists$ a degree 5 poly. which is not solvable by radicals.

Def: A finite gp. G is <u>solvable</u> (UK: "soluable") if

$$\{1\} = G_s \trianglelefteq G_{s-1} \trianglelefteq \cdots \trianglelefteq G_0 = G$$

where G_i/G_{i+1} is cyclic. (abelian also works)

Examples:

- abelian gps.
- dihedral gps.

e.g. $\mathrm{Gal}(K/\mathbb{Q})$ where $K = \mathbb{Q}(\sqrt{D})$ or $K = \mathbb{Q}(\zeta_n)$

$$1 \trianglelefteq C_n \trianglelefteq D_{2n}$$

C_2

- p-gps. $(|G| = p^k)$

- S_4: $1 \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4$

V_4 C_2 C_2

Non-examples:

- $\underbrace{S_n \text{ or } A_n \text{ for } n \geq 5}_{\text{no normal subgps: ie. "simple"}}$ (D&F Thm 4.24)

- Other finite simple gps. (e.g. the monster)

Cor: If $n = 5$, $K = Sp_F f$,

$\mathrm{Gal}(K/F) = S_n$ or $A_n \Rightarrow f$ is not solvable by radicals

So Galois $\Rightarrow$ Abel-Ruffini

Prop:

a) If $H \leq G$, then G solvable $\Rightarrow$ H solvable

b) If $H \trianglelefteq G$, then H solvable, G/H solvable $\Rightarrow$ G solvable

Pf:

a) Let $\{1\} = G_s \triangleleft G_{s-1} \triangleleft \cdots \triangleleft G_0 = G$

where G_i / G_{i+1} is cyclic, and let $H_i = H \cap G_i$

Then $H_{i+1} \triangleleft H_i$ and H_{i+1}/H_i is isom to a subgp. of G_{i+1}/G_i, so is cyclic.

b) $$1 = H_s \triangleleft H_{s-1} \triangleleft \cdots \triangleleft H_0 = H$$

$$1 = J_r \triangleleft J_{r-1} \triangleleft \cdots \triangleleft J_r = G/H$$

If $\pi : G \to G/H$, then

$$1 = H_s \triangleleft \cdots \triangleleft H_0 = \pi^{-1}(J_r) \triangleleft \pi^{-1}(J_{r-1}) \triangleleft \cdots \triangleleft \pi^{-1}(J_1) = G$$

cyclic

$\square$

Example: $K = Sp_{\mathbb{Q}}(x^3-2)$

$$\begin{array}{ccc}
K & & 1 \\
3 \big| & & 3 \big| \\
\mathbb{Q}(\zeta_3) & \leftrightarrow & C_3 \\
2 \big| & & 2 \big| \\
\mathbb{Q} & & S_3
\end{array}$$

$$\mathrm{Gal}(K/K) \trianglelefteq \mathrm{Gal}(K/\mathbb{Q}(\zeta_3)) \trianglelefteq \mathrm{Gal}(K/\mathbb{Q})$$

$$1 \trianglelefteq C_3 \trianglelefteq S_3$$

Lemma: If $F \subseteq E \subseteq K$ w/ K/F, E/F Galois, then $\mathrm{Gal}(K/E), \mathrm{Gal}(E/F)$ solvable $\Rightarrow \mathrm{Gal}(K/F)$ solvable

Pf: Since E/F Galois, by Property 4 of the Fun. Thm., $\mathrm{Gal}(K/E) \trianglelefteq \mathrm{Gal}(K/F)$ and $\mathrm{Gal}(E/F) \cong \mathrm{Gal}(K/F) / \mathrm{Gal}(K/E)$

By part b of earlier prop, $\mathrm{Gal}(K/F)$ is solvable. $\square$

From now on, we'll work in char 0

Remark: Galois gps. of ext'ns of finite fields are always cyclic, so always solvable by radicals (just take a finite field of the correct degree).

Lemma: Let char $F=0$. If $a\in F$, $K=Sp_F\, x^n-a$, then $Gal(K/F)$ is solvable.

Pf: K is the splitting field of a sep. poly, so K/F is Galois. In particular, if α is a root of x^n-a, then the roots are

$$\{\alpha\zeta_n^k \mid 0\le k<n\}$$

Let $E=F(\zeta_n)$. $Gal(E/F)$ is abelian since it's isom. to a subgp. of $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q})\cong(\mathbb{Z}/n\mathbb{Z})^\times$.

Furthermore, the map

$$\begin{aligned} Gal(K/E) &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ (\alpha\mapsto\alpha\zeta_n^k) &\longmapsto k \end{aligned}$$

is an inj. homom., so Gal(K/E) is cyclic. By the lemma, Gal(K/F) is solvable.

$\square$