Announcements

HW1 due Wednesday @ 9am via Gradescope

Don't be late! (see syllabus) (entry code: VB7EY2)

HW2 will be posted soon (due next Wed.)

Principal Ideal Domains

Recall: A Euclidean domain is an int. domain R w/ a norm $N: R \to \mathbb{Z}_{\geq 0}$ s.t. $N(0) = 0$ and $\forall a, b \in R$, $b \neq 0$, $\exists q, r \in R$ with $a = qb + r$ and $r = 0$ or $N(r) < N(b)$

Def: A principal ideal domain (PID) is an integral domain in which every ideal is principal.

Last time: Euclidean domain $\Rightarrow$ PID

Next time: PID $\Rightarrow$ "unique factorization domain" (UFD)

Def/recall:

$$R[a] = \{ r_0 + r_1 a + r_2 a^2 + \cdots + r_n a^n \mid r_i \in R, n \in \mathbb{Z}_{\geq 0} \} / \text{equiv.}$$

Integral domain

HW 1

$\mathbb{Z}[\sqrt{-5}]$ $\mathbb{Z}[\sqrt{-3}]$

$\mathbb{Z}[\sqrt{-5}][x]$ ← lecture 5

UFD

$F[x,y]$ $\mathbb{Z}[x]$

F: field

lecture 3 & lecture 5
(not PID) (UFD)

PID

$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$

D&F
p. 282

ED $\mathbb{Z}$ $\mathbb{Z}[i]$

lecture 2

F $F[x]$

Prop: R : PID. Let $a, b \in R$, $(a,b) = (d)$.
Then

a) $d = sa + tb$ for some $s, t \in R$

b) d is a gcd of a and b

Pf: a) is a consequence of $d \in (d) \subseteq (a,b) = \{sa+tb\}$.

b) Since $a, b \in d$, d is a common divisor of a & b.
If $d' | a$, $d' | b$, then $d' | sa + tb = d$, so d is a gcd of a & b. $\square$

Remark: Consider $F[x,y]$: not a PID since (x,y) is not principal. We have $1 = \gcd(x,y)$, but can't have $1 = sx + ty$.

Recall/Def: Let $r \in R$: integral domain

a) r is a unit if $\exists s \in R$ w/ $rs = sr = 1$

If r not unit, $r \neq 0$

b) r is irreducible if $r = ab \Rightarrow a$ or b is a unit

c) r is prime if $r | ab \Rightarrow r | a$ or $r | b$

Prop: r is prime $\Rightarrow$ r is irreducible

Pf: Let $r=ab$, and assume WLOG that $a=rt$.
Then $r=ab=rtb \Rightarrow r(1-tb)=0 \Rightarrow tb=1 \Rightarrow b$ is a unit.

$\square$

Converse doesn't hold: 3 is irred. in $\mathbb{Z}[\sqrt{-5}]$, but $3^2=9=(2+\sqrt{-5})(2-\sqrt{-5})$ and $3 \nmid 2\pm\sqrt{-5}$, so 3 is not prime.

Recall/Def: Let I be an ideal in R

a) I is maximal if either/both:

- $\nexists$ ideal J s.t. $I \subsetneq J \subsetneq R$
- R/I is a field

b) I is prime if either/both:

- $ab \in I \Rightarrow a \in I$ or $b \in I$
- R/I is an integral domain

So maximal $\Rightarrow$ prime

Lemma: If $r \neq 0$, (r) prime ideal $\Leftrightarrow$ r prime elt.

Pf: $a \in (r) \Leftrightarrow a$ is a multiple of r. So,

$$\underbrace{[ab \in (r) \Rightarrow a \in (r) \text{ or } b \in (r)]}_{\text{prime ideal}} \Leftrightarrow \underbrace{[r \mid ab \Rightarrow r \mid a \text{ or } r \mid b]}_{\text{prime elt.}}$$

Prop: Every nonzero prime ideal in a PID is maximal

Pf: Let $0 \subsetneq (p) \subseteq (m) \subsetneq R$, (p): prime.

By the previous results, (p) prime $\Rightarrow p$ prime $\Rightarrow p$ irred.

Since $(p) \subseteq (m)$, $p = am$ for some a, so either

- a is a unit $\Rightarrow (m) = (p)$
- m is a unit $\Rightarrow (m) = R$

Therefore, (p) is maximal. $\square$

Cor: If $r \in R$: PID, r prime $\Leftrightarrow$ r irred.

Pf: $\Rightarrow$ holds in any int. dom. (see earlier)

$\Leftarrow$: By previous pf,

r irred. $\Rightarrow (r)$ maximal $\Rightarrow (r)$ prime $\Rightarrow r$ prime. $\square$

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ED

$\mathbb{Z}$ $\qquad$ $\mathbb{Z}[i]$

lecture 2

F $\qquad$ $F[x]$

Ex: $\mathbb{Z}[x]$ is not a PID since $(2, x)$ is not principal

Prop: $R[x]$: PID $\Leftrightarrow$ R: field

Pf: $\Leftarrow$) If R: field, $R[x]$ is Euclidean (last time), hence a PID.

$\Rightarrow$)

$R[x]$ integral domain $\Rightarrow$ R integral domain

$\Rightarrow (x)$ prime (since $R[x]/(x) \cong R$)

$\Rightarrow (x)$ maximal (since it is a prime ideal in a PID)

$\Rightarrow R \cong R[x]/(x)$ field $\square$