Announcements:

Midterm exams etc. now scheduled (see website or email)

Office hours: Mon/Fri after class

Problem sessions: Tues. 3:00-4:20pm

Join Gradescope course if you haven't already!

(entry code: VB7EY2)

Euclidean Domains

Unless otherwise stated, all rings are commutative and have 1.

Def: An (integral) domain is a (commutative, nonzero) ring w/out zero divisors: if $a \neq 0$, $b \neq 0$, then $ab \neq 0$.

Def:

a) A norm is a function $N: R \to \mathbb{Z}_{\geq 0}$ with $N(0) = 0$

b) N is Euclidean if $\forall a, b \in R$, $b \neq 0$, $\exists q, r \in R$ s.t.

- $a = qb + r$
- $r = 0$ or $N(r) < N(b)$

quotient remainder

c) A Euclidean domain is an int. domain w/ a Euclidean norm

Idea: we can use the Euclidean algorithm to find gcds

Ex: a) $\mathbb{Z}$ w/ $N(a) = |a|$

b) F: field w/ $N(a) = 0$

c) $F[x]$ (F: field) w/ $N(p(x)) = \deg p$

d) $\mathbb{Z}[i]$ w/ $N(a+bi) = |a+bi|^2 = a^2+b^2$ ← later today

Non-ex: $\mathbb{Z}[\sqrt{-5}]$ (next week)

Def:

a) Write $a|b$ (in R) if $\exists x \in R$ s.t. $ax = b$

"a divides b"

b) $d \in R$ is a gcd of a and b if

- $d|a$ and $d|b$
- If $d'|a$ and $d'|b$, then $d'|d$

(gcd is always unique up to units)

Thm: Let R: Euclidean domain, $a, b \in R$, $b \neq 0$. Then a and b have a gcd.

Pf: Apply Euclidean algorithm:

$$a = q_0 b + r_0 \qquad \begin{pmatrix} a = r_{-2} \\ b = r_{-1} \end{pmatrix}$$

$$b = q_1 r_0 + r_1$$

$r_0 = q_2 r_1 + r_2$

$\vdots$

$r_{n-1} = q_{n+1} r_n + \underbrace{0}_{= r_{n+1}}$

where

$N(b) > N(r_0) > \cdots > N(r_n)$

At each step, notice that

d is a common divisor of r_i and r_{i+1} $\Leftrightarrow$ d is a common divisor of r_{i+1} and r_{i+2}

So gcds are unchanged at each step There fore, the gcd of a and b equals $\gcd(r_n, 0) = r_n$. $\square$

Recall (from D&F Ch.7):

a) An <u>ideal</u> $I \subseteq R$ is an additive subgp. s.t. if $a \in I$, $r \in R$, then $ra \in I$.

b) I is <u>principal</u> if I has the form

$$(a) := \{ ra \mid r \in R \}$$

Thm: If R: Euclidean domain, then every ideal is principal.

Pf: Choose $d \neq 0$ in I w/ minimum norm. If $a \in I$, then by the Euclidean property

$$a = qd + r$$

w/ $r = 0$ OR $\underbrace{r \neq 0 \text{ and } N(r) < N(d)}_{\text{impossible by assumption}}$

Thus, $d \mid a$, so $I = (d)$. $\square$

Pf that $\mathbb{Z}[i]$ is a Euclidean domain:

Let $a, b \in \mathbb{Z}[i]$.

Let q be an element of $\mathbb{Z}[i]$ closest to a/b (in $\mathbb{C}$). (i.e. $|q - a/b|$ is minimal)

Let $r = a - qb \in R$ (so that $a = qb + r$)

We have

$$N(r) = |r|^2 = |a - qb|^2 = \underbrace{\left|\frac{a}{b} - q\right|}_{\leq \frac{1}{\sqrt{2}}}^2 |b|^2 \leq \frac{1}{2}|b|^2 < N(b) \quad \square$$