Announcements:

Midterm course feedback form (see email)

https://forms.gle/xgQWQZneC7UBsLgV6

Finite fields

Prop: Let $n > 0$, p: prime. There exists a finite field w/ p^n elts., unique up to isom.

Pf: Existance

$$\text{Let } f(x) := x^{p^n} - x \in \mathbb{F}_p, \quad F := Sp_{\mathbb{F}_p}(f) =: \mathbb{F}_{p^n}$$

Since $\mathbb{F}_p$ is sep., f has p^n distinct roots in F and such a root α satisfies $\alpha^{p^n} = \alpha$

These roots form a subfield of F:

$$(\alpha\beta)^{p^n} = \alpha^{p^n}\beta^{p^n} = \alpha\beta, \quad (\alpha^{-1})^{p^n} = (\alpha^{p^n})^{-1} = \alpha^{-1},$$

$$(\alpha+\beta)^{p^n} = \underbrace{\mathrm{Frob}(\cdots(\mathrm{Frob}}_{n}(\alpha+\beta)\cdots)$$

$$= \mathrm{Frob}(\cdots(\mathrm{Frob}(\alpha)\cdots) + \mathrm{Frob}(\cdots(\mathrm{Frob}(\beta)\cdots)$$

$$= \alpha^{p^n} + \beta^{p^n}$$

So by minimality, $F = \{\text{roots of } x^{p^n} - x\}$

$|F| = p^n, \qquad [F : \mathbb{F}_p] = n$

Let K be any field of order p^n. Then char $K = p$, $[K : \mathbb{F}_p] = n$.

We have $|K^\times| = |K| - 1 = p^n - 1$, so if $\alpha \in K$, $\alpha^{p^n - 1} = 1$, so $\alpha^{p^n} = \alpha$, α is a root of $x^{p^n} - x$.

Since K has $|K| = p^n$ roots of this poly, it is the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$, which is unique up to isom. $\square$

Cyclotomic Fields

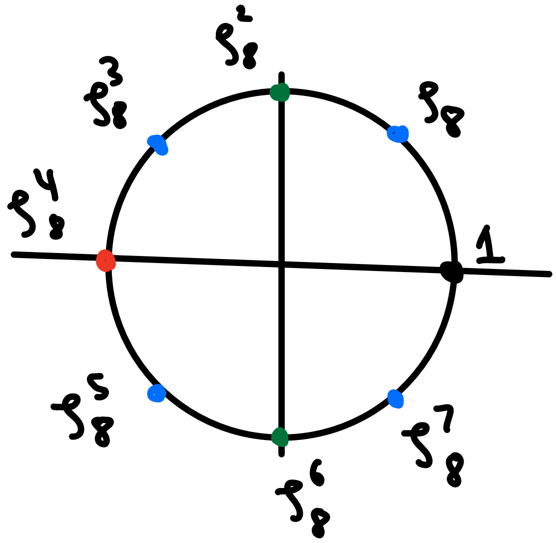
$\mathbb{Q}(\zeta_n)$ where $\zeta_n = e^{2\pi i/n}$

$$\mu_n = \left\{ \begin{array}{c} \text{all } n\text{th roots} \\ \text{of } 1 \text{ in } \mathbb{C} \end{array} \right\} = \{1, \zeta_n, \ldots, \zeta_n^{n-1}\} = \langle \zeta_n \rangle \subseteq \mathbb{Q}(\zeta_n)$$

Primitive nth root: a generator ζ of μ_n i.e.

$\zeta^d \neq 1$ for $d < n$.

Which ζ_n^k are primitive?



primitive...

- 1st roots of 1
- 2nd roots of 1
- 4th roots of 1
- 8th roots of 1

$$\mu_n \xrightarrow[\text{gp. isom.}]{\sim} \mathbb{Z}/n\mathbb{Z}$$

under mult.

under +

$$\zeta_n^k \longmapsto k$$

So ζ_n^k primitive $\Leftrightarrow \gcd(k,n) = 1$

Euler φ function: $\varphi(n) = |\{0 < k < n \mid \gcd(k,n) = 1\}|$

$= |\{\text{prim. } n\text{th roots of } 1\}|$

$\varphi(p) = p-1$ $\qquad$ $\varphi(ab) = \varphi(a)\varphi(b)$ if $\gcd(a,b)=1$

p : prime

$\varphi(p^k) = p^{k-1}\cdot(p-1)$ $\qquad$ Thus, $\varphi(p_1^{k_1}\cdots p_n^{k_n}) = \prod_{i=1}^{n} p_i^{k_i-1}(p_i-1)$

Def: The cyclotomic polynomial is

$$\Phi_n(x) = \prod_{\substack{\zeta\in\mu_n \\ \text{prim.}}} (x-\zeta) = \prod_{\substack{0\le k<n \\ \gcd(k,n)=1}} (x-\zeta_n^k)$$

E.g.:

$\Phi_1 = x-1$ $\qquad$ $\Phi_4 = x^2-1$

$\Phi_2 = x+1$ $\qquad$ $\Phi_5 = x^4+x^3+x^2+x+1$

$\Phi_3 = x^2+x+1$ $\qquad$ $\Phi_6 = x^2-x+1$

$$x^n-1 = \prod_{\zeta\in\mu_n}(x-\zeta) = \prod_{d|n}\left(\prod_{\substack{\zeta\in\mu_d \\ \text{prim.}}}(x-\zeta)\right) = \prod_{d|n}\Phi_d(x)$$

Facts:

a) $\Phi_d(x) \mid x^n - 1$ if $d \mid n$ (or if $d=n$)

b) Every root ζ of unity is a root of precisely one Φ_n

c) $\deg \Phi_n = \varphi(n)$

d) Φ_n is monic

Thm: $\Phi_n(x) \in \mathbb{Z}[x]$ and is irred. (over $\mathbb{Z}$ or $\mathbb{Q}$)

Cor:

a) $m_{\zeta_n, \mathbb{Q}} = \Phi_n(x)$

b) $[\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)$

Pf of Thm:

$\Phi_n \in \mathbb{Z}[x]$: Induction on n ($n=1$: clear)

Assume that $\Phi_d(x) \in \mathbb{Z}[x]$ for $d<n$

Then $x^n - 1 = f(x)\Phi_n(x)$ where $f(x) = \prod_{\substack{d \mid n \\ d<n}} \Phi_d(x)$

Divide w/ remainder in $\mathbb{Q}[x]$ since $x^n-1, f(x) \in \mathbb{Q}[x]$

$$x^n - 1 = q(x) f(x) + r(x)$$

w/ $q, r \in \mathbb{Q}[x]$, $\deg r < \deg f$

Then in $\mathbb{C}[x]$, we have

$$\Phi_n(x) f(x) = q(x) f(x) + r(x) \Rightarrow (\Phi_n(x) - q(x)) f(x) = r(x)$$

$\Rightarrow r(x) = 0$ as $\deg r < \deg f$. Thus, $\Phi_n(x) = q(x) \in \mathbb{Q}[x]$,

and by Gauss' Lemma since $x^n-1, f(x) \in \mathbb{Z}[x]$, $\Phi_n \in \mathbb{Z}[x]$ too.

Irreducible: Suppose not!

$\Phi_n(x) = f(x) g(x)$ $\quad$ f, g monic in $\mathbb{Z}[x]$, f irred.

Claim: Let ζ be a root of f. Then ζ^p is a root of f for any prime p coprime to n

Claim $\Rightarrow$ result: Iterating the claim, ζ^m is a root of f for any m coprime to n, so all prim nth roots of 1 are roots of $f \Rightarrow f = \Phi_n$.

Pf of claim: Suppose instead that $g(\zeta^p)=0$.

Then ζ is a root of $g(x^p)$, so

$$g(x^p) = f(x)h(x) \text{ for some } h(x) \in \mathbb{Z}[x]$$

Reduce mod p: $\mathbb{Z}[x] \Rightarrow \mathbb{F}_p[x]$

1) $x^n - 1$ is sep. in $\mathbb{F}_p[x]$ as $nx^{n-1} \neq 0$,

so $\overline{\Phi}_n(x)$ has distinct roots.

2) Frob: $\mathbb{F}_p \to \mathbb{F}_p$ is the identity

$(a \in \mathbb{F}_p^* \Rightarrow |a| \mid p-1 \Rightarrow a^{p-1} = 1 \Rightarrow a^p = a)$

"Fermat's Little Theorem"

Hence,

$$(\overline{g}(x))^p = \overline{g}(x^p) = \overline{f}(x)\overline{h}(x) \in \mathbb{F}_p[x]$$

3) This means that $\overline{g}$ and $\overline{f}$ have a common root

4) But then $\overline{\Phi}_n = \overline{g}\overline{f}$ has a mult. root, a contradiction

$\square$